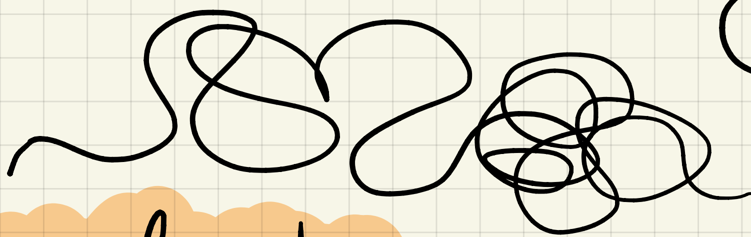
4 ways to select



n things

select k

- Is order relevant?
- Can I repeat choices?

order? — No / Yes

repetition? — No / Yes / No / Yes

unordered no repeats "k-combinations"	unordered repeats	ordered no repeats "k-permutations"	ordered repeats

Some examples we have seen so far:

- \# unordered pairs ($k=2$, no order, no repetition)

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

- \# ordered pairs ($k=2$, ordered, no repetition)

$$n(n-1)$$

- \# permutations ($k=n$, ordered, no repetition)

$n!$ (select n from n with order)

- when $k=1$, the answer is n in all 4 cases

(select 1 from n in n ways regardless)

Finding # k-permutations

Recall :

\# permutations on n objects .... $n!$

Think of task for generating one permutation

		# ways
1. choose an object for 1^{st} position	....	n
2. choose another object for 2^{nd} position	....	$n-1$
⋮		
k. choose another object for k^{th} postion	....	$n-k+1$

$n(n-1)\dots(n-k+1)$

stop

⋮

n. choose another object for n^{th} position 1

We can't permute choices in phases and get same outcome

$\Rightarrow$ No overcounting

k-permutations

$$P^n_k = {}_nP_k = P(n,k) = \overbrace{n(n-1)\dots(n-k+1)}^{k} = n^{\underline{k}}$$

falling power

or falling factorial

- Can we express it in terms of factorials?
- What happens if we multiply & divide by $(n-k)!$?

Try: $n(n-1)\ \dots\ (n-k+1) \times \frac{(n-k)!}{(n-k)!}$

$$= \frac{n(n-1)\dots(n-k+1)\times(n-k)(n-k-1)\dots 1}{(n-k)!} = \frac{n!}{(n-k)!}$$

Examples:

$$K=0: \quad \frac{n!}{(n-k)!} = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$$

$$K=1: \quad \frac{n!}{(n-k)!} = \frac{n!}{(n-1)!} = \frac{n \times (n-1)!}{(n-1)!} = n$$

$$\underbrace{K=2:}_{\text{ordered pairs}} \quad \frac{n!}{(n-k)!} = \frac{n!}{(n-2)!} = \frac{n \times (n-1) \times (n-2)!}{(n-2)!} = n(n-1)$$

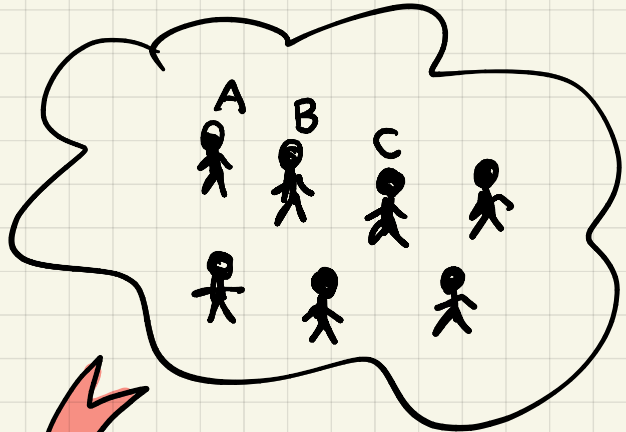
$$\underbrace{K=n:}_{\text{permutations}} \quad \frac{n!}{(n-k)!} = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{\prod_{i=1}^{0} i} = \frac{n!}{1} = n!$$

Finding # k-combinations

Recall : # k-permutations on n objects

	# ways
1. choose an object for 1^{st} position	n
2. choose another object for 2^{nd} position	$n-1$
$\vdots$	
k. choose another object for k^{th} postion	$n-k+1$
	$n!/(n-k)!$

By how much do we overcount ?



$K = 3$ (with order)

Among the possibilities:

$$\left.\begin{array}{l} A\ B\ C \\ A\ C\ B \\ B\ A\ C \\ B\ C\ A \\ C\ A\ B \\ C\ B\ A \end{array}\right\} \text{3! 3-permutations are the same 3-combination}$$

In general, K-permutations overcount K-combinations by K!

k-combinations "n choose k"

$$\binom{n}{k} = {}_nC_k = C^n_k = \frac{n!}{k!\,(n-k)!}$$

Examples:

$k=0$: $\binom{n}{0} = \dfrac{n!}{0!\,(n-0)!} = 1$

$k=1$: $\binom{n}{1} = \dfrac{n!}{1!\,(n-1)!} = n$

$k=2$: $\binom{n}{2} = \dfrac{n!}{2!\,(n-2)!} = \dfrac{n(n-1)}{2}$ [unordered pairs]

$k=n$: $\binom{n}{n} = \dfrac{n!}{n!\,(n-n)!} = 1$

What if we allow repetition?

Select k from n with order & repetition

	# ways
1. choose an object for 1^{st} position	n
2. choose an object for 2^{nd} position	n
⋮	
k. choose an object for k^{th} postion	n
	n^k

No overcount

Summary of results

Select k from n	ordered	unordered
no repetition	$\dfrac{n!}{(n-k)!}$	$\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$
repetition	n^k	? later

Examples:

How many 3-letter words can I make if

- Letters cannot repeat

$$\frac{26!}{(26-3)!} = 26 \times 25 \times 24$$

- Letters can repeat

$$26^3 \quad (\text{word size} : \underset{k}{3}, \text{ Alphabet size} : \underset{n}{26})$$

- Letters cannot repeat but must appear in alphabetical order

This amounts to choosing 3 letters with $\underbrace{\text{"no order"}}_{\text{why?}}$

$$\binom{26}{3}$$

Examples: - How many n bit patterns are there?

choose a value for each bit from $\{0,1\}$
(repetition allowed)

2^n (word size: n, Alphabet size: 2)

- How many 10 bit patterns have exactly 3 1s?

choose 3 bits out of n to make 1s

$\binom{10}{3}$

Lesson: Don't be too literal in applying the concepts. The wording of the problem does not necessarily "mimic" the formula.

Examples: While alphabetical implies order, the solution corresponded to unordered selection.

In the n-bit problem, n does not correspond to the "n" in formula n^k.