Sets, subsets, and $\binom{n}{k}$

- A set is an <u>unordered</u> collection of "things" we call elements
- We list them between $\{\ \}$ separated by $,$, like this:

  $S = \{x, y, z\} = \{z, x, y\}$ (every elem. appears once; otherwise, multiset)
- A set can be infinite (has infinitely many elements)

  how do we list them? (Later)
- If a set S is finite, the cardinality (size) of S is the number of elements in it, and its denoted by $|S|$

  e.g. $S = \{x, y, z\}$, $|S| = 3$

Ideas & Notation

Idea	Notation	Negation
x is in S x is an element of S	$\underset{\text{elem. of } S}{x} \in \underset{\text{Set}}{S}$	$x \notin S$
T is a subset of S every elem. of T is an elem. of S	$\underset{\text{Set}}{T} \subset \underset{\text{Set}}{S}$	$T \not\subset S$
T and S are equal $T \subset S$ and $S \subset T$	$T = S$	$T \neq S$

The empty set

The empty set has no elements and is denoted by

ϕ or $\{\ \}$, $|\phi| = 0$

Given any set S, $\phi \subset S$. why?

Is every element of ϕ an element of S?

Can you find an elem. of ϕ that is not an elem. of S?

Sets can be tricky!

$$S = \{\{1,2,3\},\ 4,\ (5,6),\ \heartsuit\}$$

set of ints   integer   tuple of ints

$$1 \notin S$$

$$\{1,2,3\} \not\subset S$$

$$\{1,2,3\} \in S$$

Some Known infinite sets

$\mathbb{N} = \{1, 2, 3, \ldots\} = \{x \mid x \text{ is a positive integer}\}$

"such that" or $\{x : \ldots\}$

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

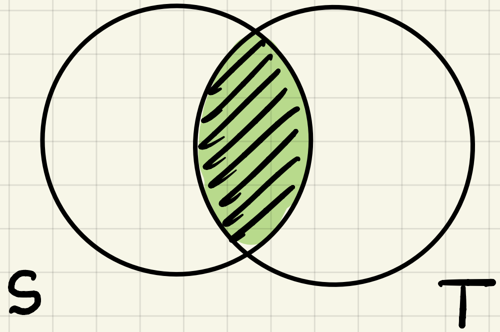
$= \{x \mid x \text{ is an integer}\}$

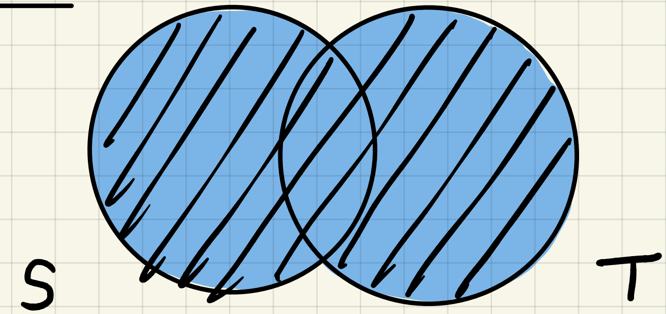
$\mathbb{Q} = \{x \mid x = \frac{a}{b} \text{ where } a \in \mathbb{Z} \text{ and } b \in \mathbb{N}\}$

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$

natural numbers   integers   rational numbers

Intersection & Union





$S \cap T = \{x \mid x \in S \text{ and } x \in T\}$

Intersection

$S \cup T = \{x \mid x \in S \text{ or } x \in T\}$

Union

not exclusive

Can be generalized to multiple sets

$$S_1 \cap S_2 \cap \ldots \cap S_n = \bigcap_{i=1}^{n} S_i = \{x \mid x \in S_1 \text{ and} \ldots \text{and } x \in S_n\}$$

$$S_1 \cup S_2 \cup \ldots \cup S_n = \bigcup_{i=1}^{n} S_i = \{x \mid x \in S_1 \text{ or} \ldots \text{or } x \in S_n\}$$

Addition rule (finite sets) :

$$|S \cup T| = |S| + |T| \text{ if } S \cap T = \emptyset$$

Disjoint

Product of Sets

$$S \times T = \{(x,y) \mid x \in S \text{ and } y \in T\}$$

(x,y) ordered

tuple

$(x,y) \neq (y,x)$

$\{x,y\} = \{y,x\}$

Example: $S = \{a,b,c\}$ $T = \{1,2\}$

$$S \times T = \{(a,1),(a,2),(b,1),(b,2),(c,1),(c,2)\}$$

$$S \times T \neq T \times S$$

Product rule (finite sets):

$$|S \times T| = |T \times S| = |S| \times |T|$$

To generate a tuple

	#ways
1. choose an elem. from S	$\lvert S\rvert$
2. choose an elem. from T	$\lvert T\rvert$
	$\lvert S\rvert \times \lvert T\rvert$

Example:

$$S = \{a, b, c\}$$

$$T = \{1, 2\}$$

$$R = \{\heartsuit, \diamondsuit\}$$

$$S \times T \times R = \{(x, y, z) \mid x \in S \text{ and } y \in T \text{ and } z \in R\}$$

$$\begin{aligned} S \times T \times R = \{ & (a,1,\heartsuit), (a,1,\diamondsuit), (a,2,\heartsuit), (a,2,\diamondsuit), \\ & (b,1,\heartsuit), (b,1,\diamondsuit), (b,2,\heartsuit), (b,2,\diamondsuit), \\ & (c,1,\heartsuit), (c,1,\diamondsuit), (c,2,\heartsuit), (c,2,\diamondsuit) \} \end{aligned}$$

$$|S \times T \times R| = |S| \times |T| \times |R| = 3 \times 2 \times 2 = 12.$$

The power set

Given a set S, the power set of S is the set of all subsets of S

$$\underbrace{\mathcal{P}(S) = 2^S}_{\text{notation}} = \{T \mid T \subset S\}$$

Example : $S = \{a, b, c\}$

$$\mathcal{P}(S) = \{\, \phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\} \,\}$$

Question : what is $|\mathcal{P}(S)|$ for finite S ?

Let's take the same example:

$$S = \{a, b, c\}$$

$$\mathcal{P}(S) = \{ \emptyset, \qquad 1 = \binom{3}{0}$$

$$\{a\}, \{b\}, \{c\}, \qquad 3 = \binom{3}{1}$$

$$\{a,b\}, \{a,c\}, \{b,c\}, \qquad 3 = \binom{3}{2}$$

$$\{a,b,c\} \} \qquad 1 = \binom{3}{3}$$

In how many ways can we choose a subset of size k?

(Remember : sets are unordered.) $\binom{n}{k}$

In how many ways can we choose k elem. from n, no order, no repetition?

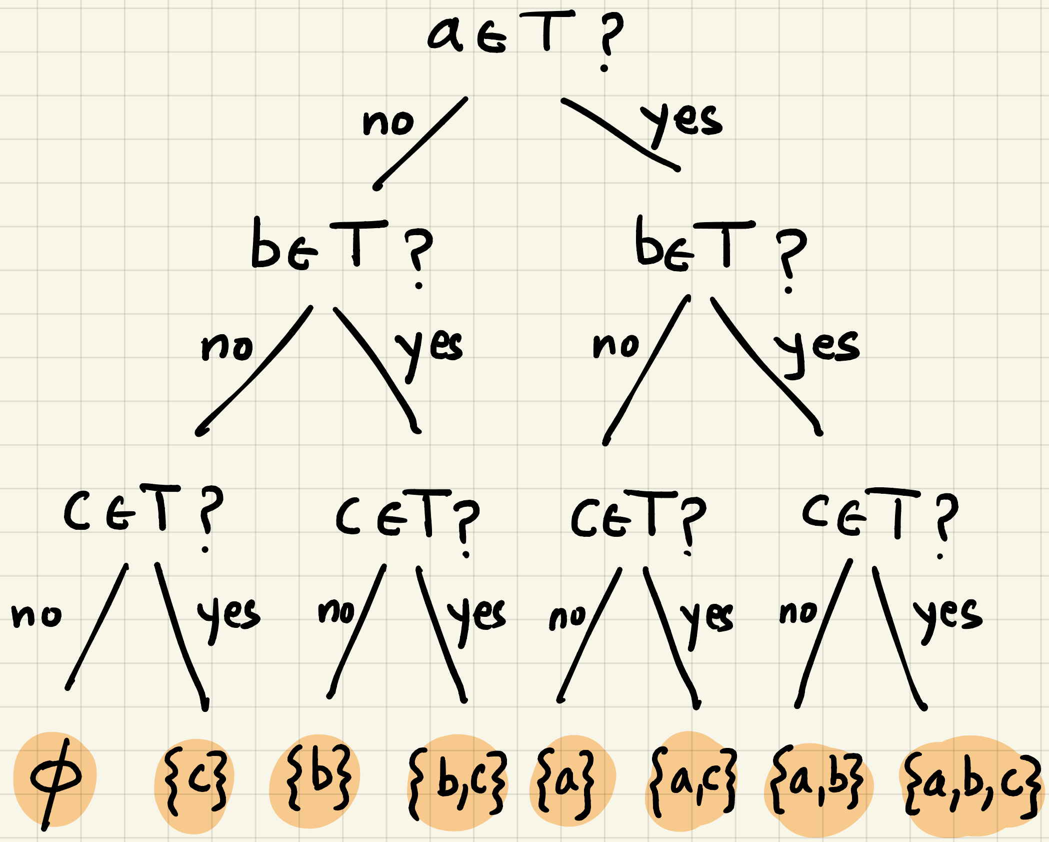
Conclusion: Given S such that $|S| = n$,

\# subsets of S of size k is $\binom{n}{s}$

So $|\mathcal{P}(S)| = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k}$

Another way to count subsets

To count the number of subsets, think about the task of generating one subset T.



The tree gets twice as big with each level: $2 \times 2 \times 2 = 8$

$S = \{a_1, a_2, \ldots, a_n\}$

	# ways
1. choose if a_1 is in subset	2
2. choose if a_2 is in subset	2
$\vdots$	
n. choose if a_n is in subset	2
	$\underbrace{2 \times 2 \times \ldots \times 2}_{n}$

Therefore, 2^n

$$\underbrace{\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots \binom{n}{n}}_{\text{These are called binomial coefficients (later)}} = 2^n \quad \text{(wow!)}$$