- Number of permutations $n!$
- Number of k-permutations $(n \geq k)$ $\frac{n!}{(n-k)!}$
  e.g. seating n people on k chairs

- Number of unordered pairs $\binom{n}{2} = \frac{n(n-1)}{2}$
- Number of k-combinations $(n \geq k)$ $\binom{n}{k}$
- e.g. # of n-bit words with k 1s

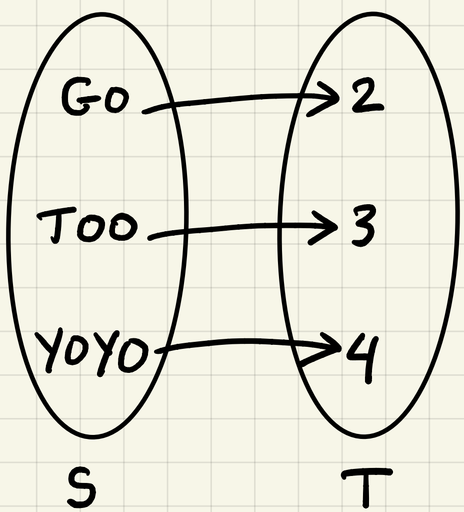
- Number of n-bit words 2^n
- Number of subsets of S, $|S| = n$ 2^n
- Number of subsets of size k $\binom{n}{k}$
- Number of k letter words, alphabet size n n^k

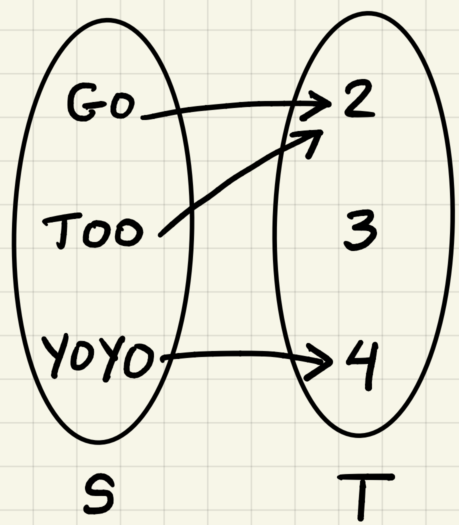
Don't just remember formulas!

Understand them, reproduce them
reason about them

Functions

- A function is a mapping from one set S into another set T
- The function "maps" elements from S to elements in T
- The function assigns to every element in S exactly one element in T

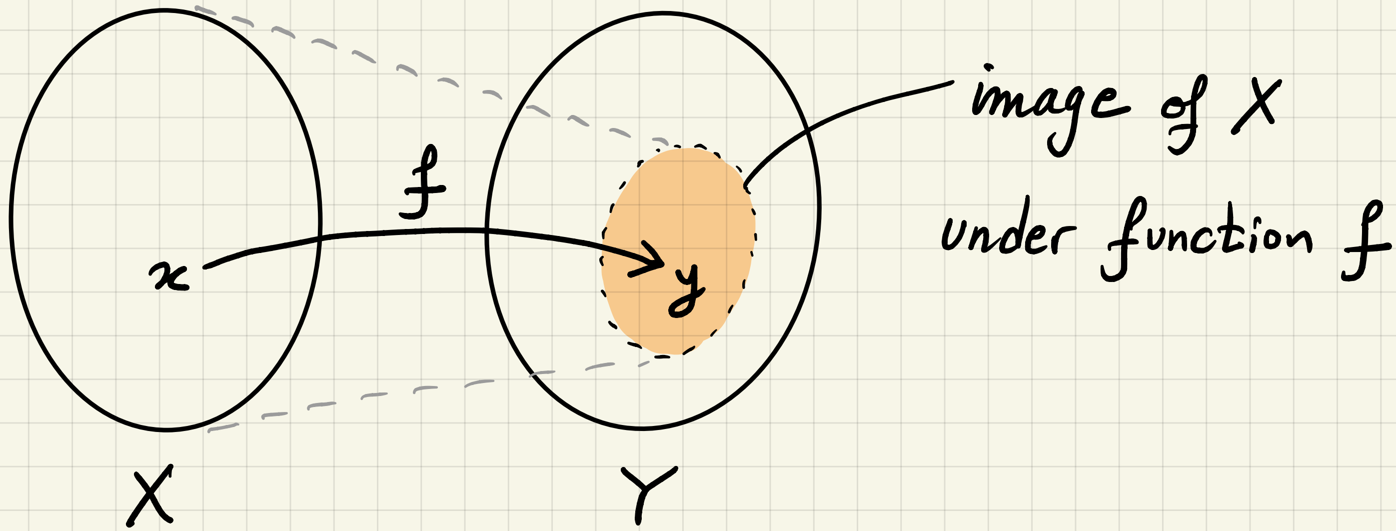




It does not have to "make sense"!

Function :

$$f : X \longrightarrow Y$$



Domain   Co-domain

$x \in X$ $y \in Y$

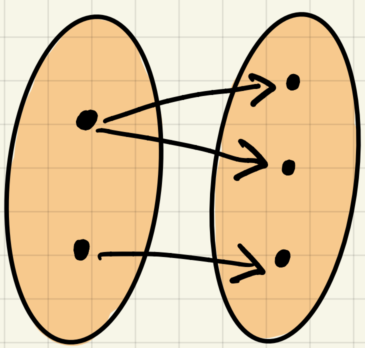
$$y = f(x)$$

When the image is the entire set Y, f is **onto** .

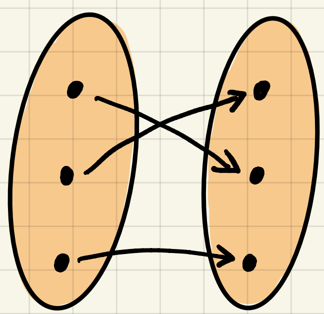
f is a function : For every $x \in X$, there exists <u>exactly one</u> $y \in Y$ such that $f(x) = y$

f is onto : For every $y \in Y$, there exists an $x \in X$ such that $f(x) = y$

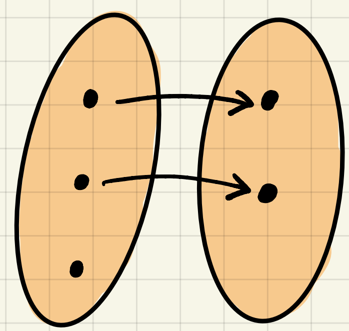
Examples: Which of the following is a function ?



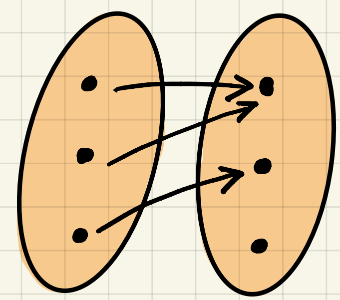
✗



✓ onto



✗



✓ not onto

Which is an onto function ?

Some more notations:

$\forall$: universall quantifier "All"

$\exists$: existential quantifier "Exists"

$\exists!$: unique existential quantifier "Exists exactly one"

function : $\forall x \in X, \exists! y \in Y, f(x) = y$

onto : $\forall y \in Y, \exists x \in X, f(x) = y$

(Say them in English)

Another property of functions

one-to-one : For every $x_1, x_2 \in X$,

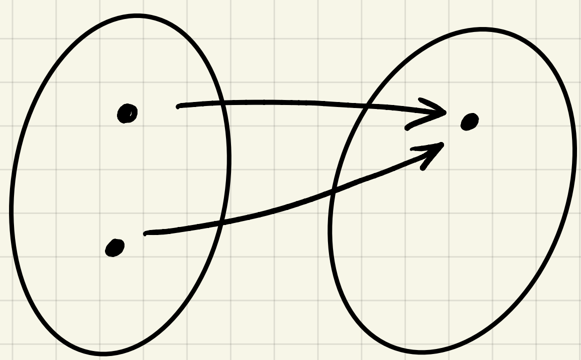
if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$

$$\forall x_1 \in X,\ \forall x_2 \in X,\ (x_1 \neq x_2) \Rightarrow f(x_1) \neq f(x_2)$$

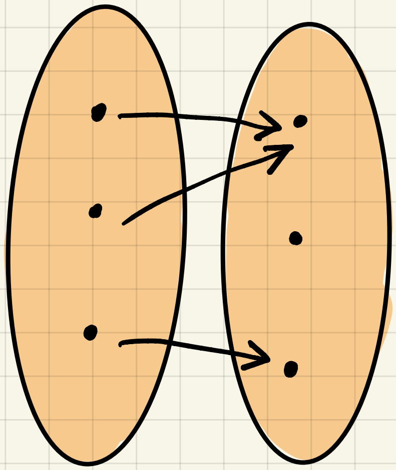
or

$$\forall x_1, x_2 \in X,\ (x_1 \neq x_2) \Rightarrow f(x_1) \neq f(x_2)$$

We can't have :

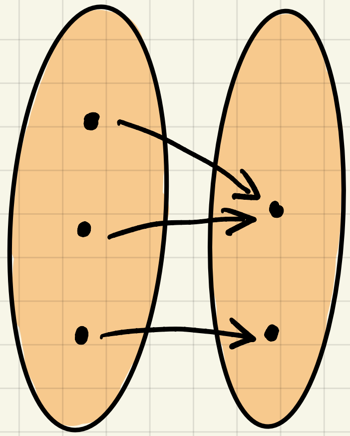


Not one-to-one



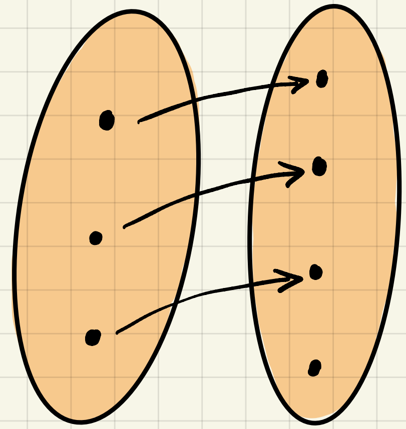
One-to-one : No

onto : No



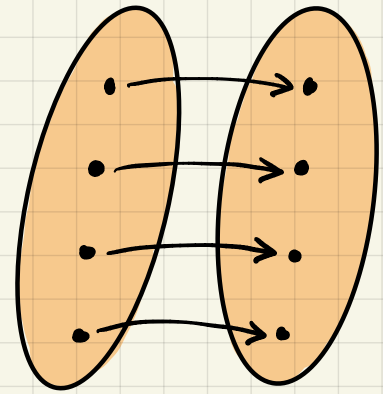
One-to-one : No

onto : Yes



One-to-one : Yes

onto : No



$\underbrace{\text{One-to-one : Yes} \quad \text{onto : Yes}}_{\text{Bijection}}$

If $f : X \to Y$ is a bijection

then $|X| = |Y|$ (useful for counting)

Other terminology

one-to-one	injection
onto	surjection
one-to-one correspondence	bijection

A bijection can be inverted

If $f: X \to Y$ is a bijection with $f(x) = y$ then there exists a bijection $g: Y \to X$ such that

$$f(x) = y \iff x = g(y) = f^{-1}(y)$$

We say that g is f^{-1} (the inverse of f)

- How to show that $f: X \to Y$ is one-to-one ?
  - ➤ show if $f(x_1) = f(x_2)$ then $x_1 = x_2$
- How to show that $f: X \to Y$ is onto ?
  - ➤ Given $y \in Y$, find an $x \in X$ such that $f(x) = y$
- How to show that $f: X \to Y$ is a bijection ?
  - ➤ Show both : one-to-one & onto .

Example: $f : (0,1) \to \mathbb{R}$

$$f(x) = \ln \frac{x}{1-x}$$

Note: $\mathbb{R}$ is the set of real numbers: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

$$(0,1) = \{ x \in \mathbb{R} \mid 0 < x < 1 \}$$

one-to-one: $f(x_1) = f(x_2) \Rightarrow \ln \frac{x_1}{1-x_1} = \ln \frac{x_2}{1-x_2}$

$$\Rightarrow \frac{x_1}{1-x_1} = \frac{x_2}{1-x_2} \Rightarrow x_1(1-x_2) = x_2(1-x_1)$$

$$\Rightarrow x_1 - x_1x_2 = x_2 - x_1x_2 \Rightarrow x_1 = x_2.$$

onto: Given $y \in \mathbb{R}$, find an $x \in (0,1)$ that satisfies $f(x) = y$, so $\ln \frac{x}{1-x} = y$

Find x:

$$\frac{x}{1-x} = e^y$$

$$x = (1-x)e^y = e^y - xe^y$$

$$x(1+e^y) = e^y$$

$$x = \frac{e^y}{1+e^y}$$

Finally, <u>make sure $x \in (0,1)$</u>: $e^y > 0 \Rightarrow x > 0$

$e^y < 1 + e^y \Rightarrow x < 1$

(both positive $\Rightarrow$ ratio < 1)

Next time : Counting with bijections !