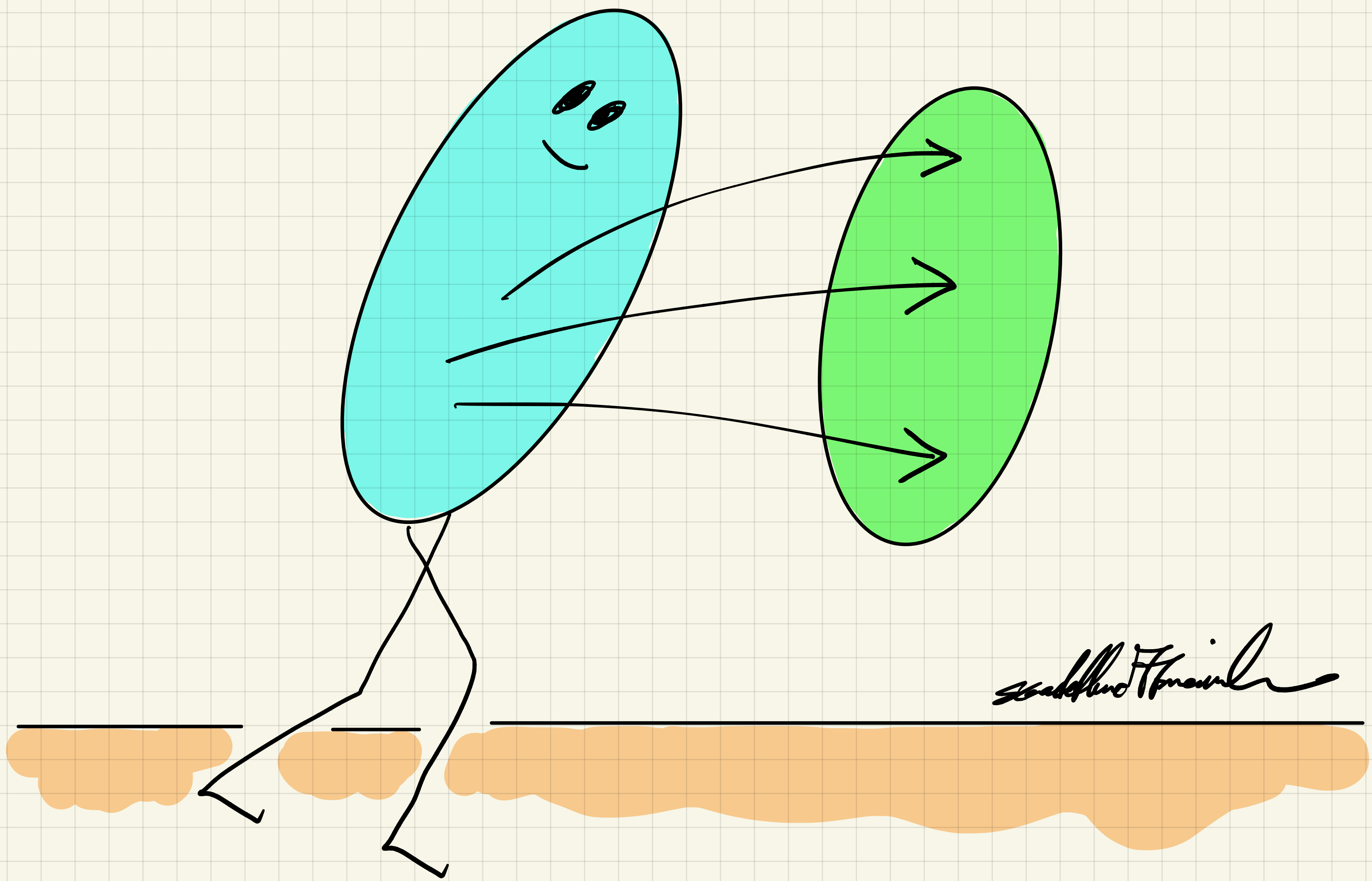
Counting with bijections!



Counting with bijections

Recall ...

$\#$ n-bit words .... 2^n

$\#$ subsets of $S = \{a_1, \dots, a_n\}$ .... 2^n

Coincidence ?

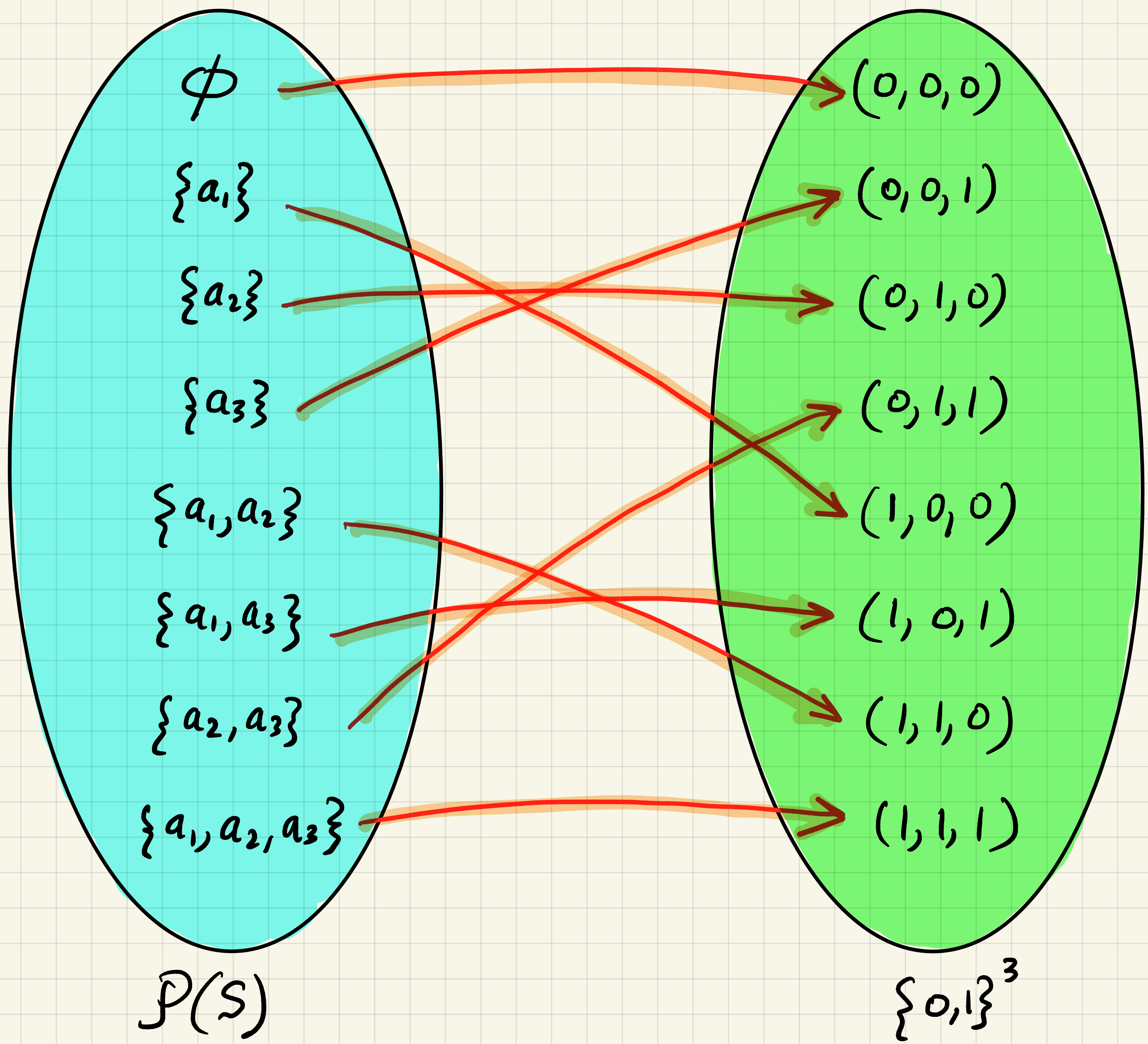
Assume you knew there are 2^n n-bit words, but nothing about the number of subsets.

Define $f : \mathcal{P}(S) \longrightarrow \{0,1\}^n = \underbrace{\{0,1\} \times \{0,1\} \times \dots \times \{0,1\}}_{n}$

$$f(T) = (b_1, b_2, \dots, b_n)$$

$$\text{where} \quad b_i = \begin{cases} 1 & \text{if } a_i \in T \\ 0 & \text{if } a_i \notin T \end{cases}$$

Example : $S = \{a_1, a_2, a_3\}$ $(n=3)$



Is $f : \mathcal{P}(S) \longrightarrow \{0,1\}^n$ as defined above a bijection ?

One-to-one : $f(T) = f(T') \Rightarrow$

$$(b_1, b_2, \dots, b_n) = (b_1', b_2', \dots, b_n')$$

- $a_i \in T \Rightarrow b_i = 1 \Rightarrow b_i' = 1 \Rightarrow a_i \in T'$, so $T \subset T'$
- $a_i \in T' \Rightarrow b_i' = 1 \Rightarrow b_i = 1 \Rightarrow a_i \in T$, so $T' \subset T$

Therefore, $T = T'$

onto: Given $(b_1, b_2, \ldots, b_n) \in \{0,1\}^n$

Construct T such that $\begin{cases} a_i \in T & \text{if } b_i = 1 \\ a_i \notin T & \text{if } b_i = 0 \end{cases}$

obviously $T \in \mathcal{P}(S)$ and $f(T) = (b_1, b_2, \ldots, b_n)$.

Therefore f is a bijection (being both one-to-one & onto)

So, $|\mathcal{P}(S)| = |\{0,1\}^n| = 2^n$ ☺

Select k out of n

Select k from n	ordered	unordered
no repetition	$\frac{n!}{(n-k)!}$	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$
repetition	n^k	?

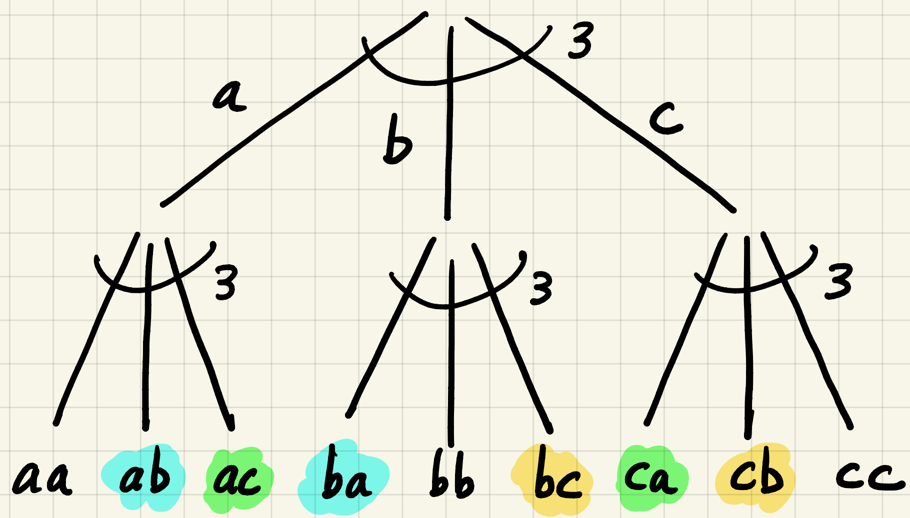
Example: $S = \{a, b, c\}$ $n = 3$

$k = 2$

aa	ab	ac
ba	bb	bc
ca	cb	cc

6 ways

Product rule does not work !



Some outcomes are overcounted, some are not

Can't adjust for overcounting

Different outcomes overcounted differently

Given $S = \{a_1, a_2, a_3\}$

Consider $^2S = \{\{a_1,a_1\}, \{a_1,a_2\}, \{a_1,a_3\}, \{a_2,a_2\}, \{a_2,a_3\}, \{a_3,a_3\}\}$

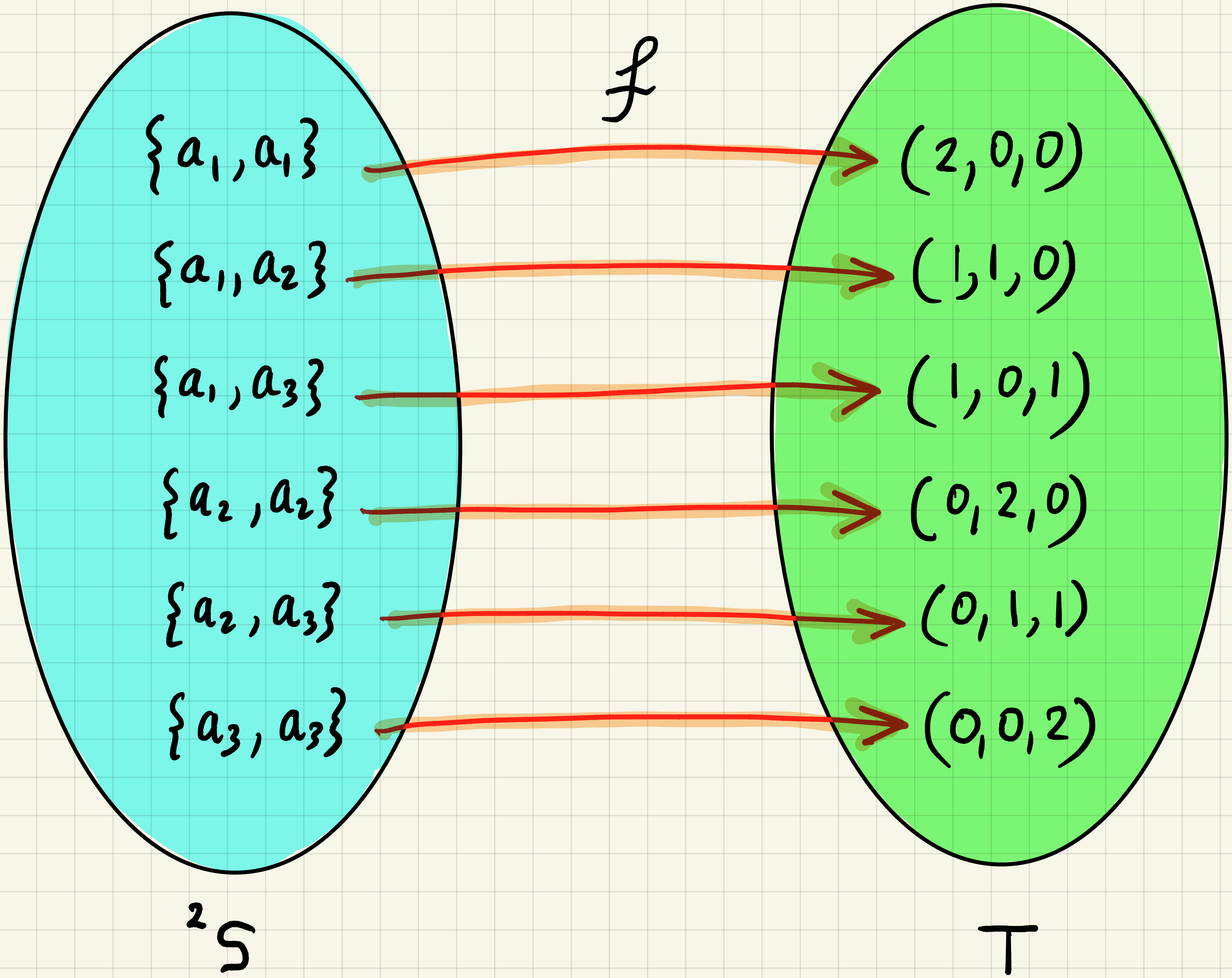
Note: This is different than $S^2 = S \times S = \{(a_1,a_1), (a_1,a_2), (a_1,a_3),$
$(a_2,a_1), (a_2,a_2), (a_2,a_3)$
$(a_3,a_1), (a_3,a_2), (a_3,a_3)\}$

$$T = \{(x_1,x_2,x_3) \mid x_1, x_2, x_3 \in \mathbb{Z}_{\geq 0}, x_1 + x_2 + x_3 = 2\}$$

$$\mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \ldots\}$$

$$f : {}^2S \to T$$

$f(s) = (x_1, x_2, x_3)$ where $x_1 = \#\, a_1$ in s
$x_2 = \#\, a_2$ in s
$x_3 = \#\, a_3$ in s



It's not hard to show f is a bijection

- If s and s' map to the same element in T, they agree on the multiplicity of all elements in $\{a_1, \ldots, a_n\}$
- Every element in T corresponds to some element in ${}^2S$

In general, we have a bijection

$$f : {}^{k}S \longrightarrow \left\{ \underbrace{(x_1, x_2, \dots, x_n)}_{n\text{-tuple}} \in \mathbb{Z}^n_{\geqslant 0} \;\middle|\; \sum_{i=1}^{n} x_i = k \right\}$$

So we have to count the number of integer solutions to:

n parts

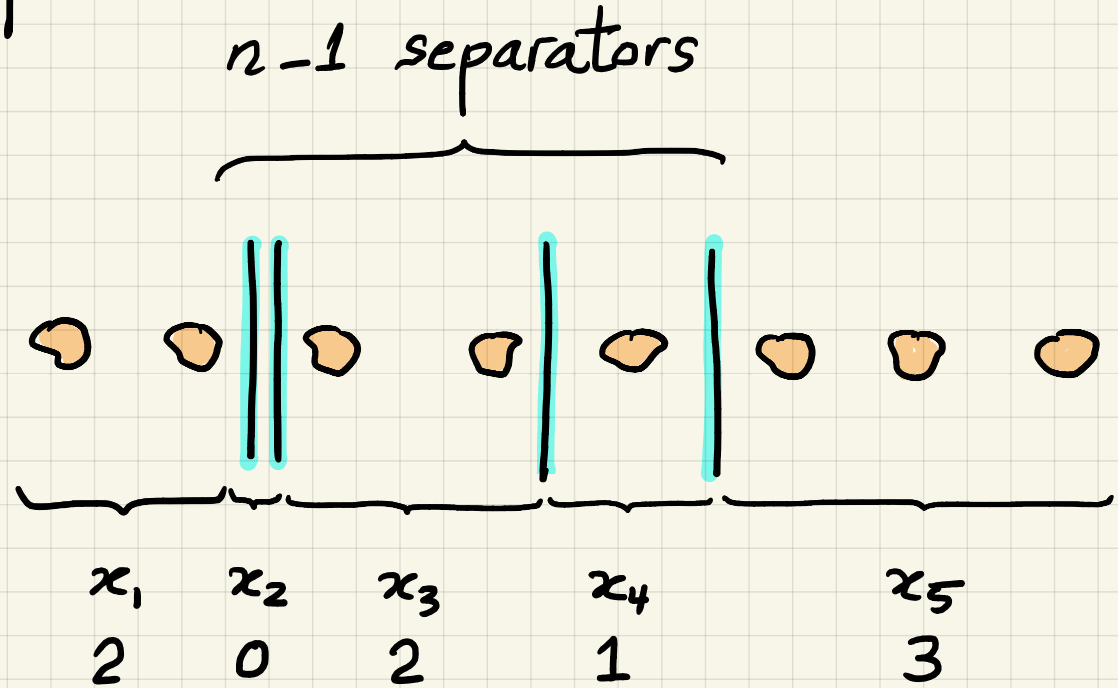
$$x_1 + x_2 + \dots + x_n = k$$

$$x_i \geqslant 0$$

This is equivalent to partitioning k into n ordered parts

This is equivalent to separating k rocks into n groups

Example:

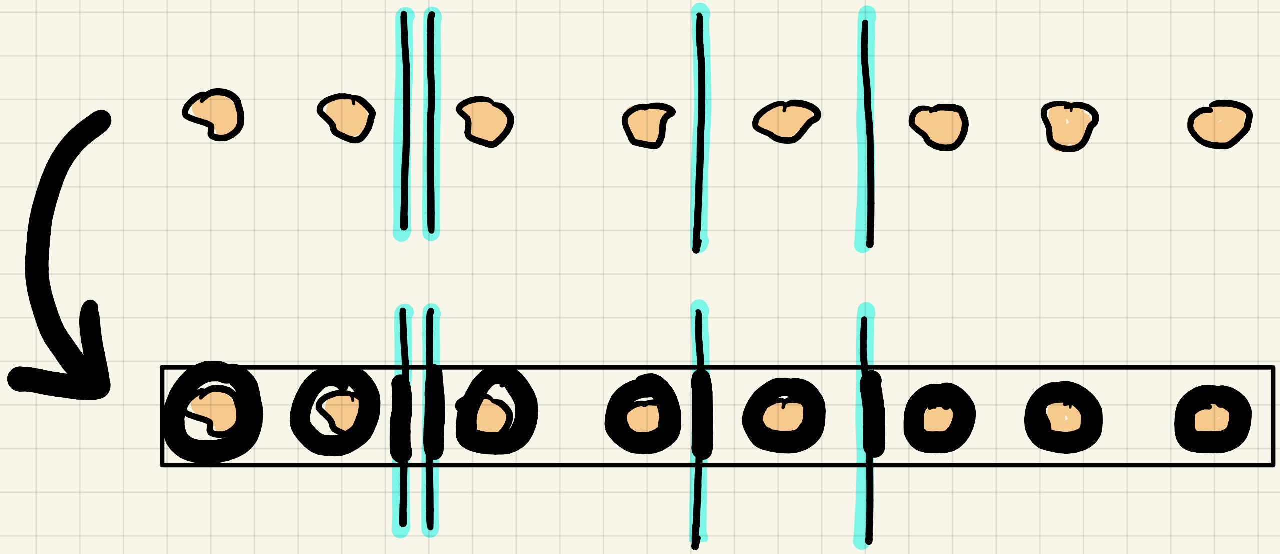


$k=8$

$n=5$

$$x_1 + x_2 + x_3 + x_4 + x_5 = 8$$

This is equivalent to making a binary word with $n-1$ 1s and k 0s



How many $(n-1+k)$-bit words have $n-1$ 1s?

$$\begin{array}{r} \#\text{ bits} \\[2em] \#\text{ 1s} \end{array} \binom{n-1+k}{n-1}$$

$n \geq k$

Select k from n	ordered	unordered
no repetition	$\dfrac{n!}{(n-k)!}$	$\dbinom{n}{k} = \dfrac{n!}{k!(n-k)!}$
repetition	n^k	$\dbinom{n-1+k}{n-1} = \underbrace{\left(\!\!\dbinom{n}{k}\!\!\right)}_{\text{Notation}}$

n choose k with rep.

<u>Example:</u> In how many ways can we select 3 elements from $\{a,b,c,d,e,f,g\}$ with repetition & no order?

Same as number of integer solutions to

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 3, \quad x_i \geq 0$$

Answer: $\left(\!\!\dbinom{7}{3}\!\!\right) = \dbinom{7-1+3}{7-1} = \dbinom{9}{6} = \dfrac{9!}{6!\,3!}$

We can handle more general $\geq$ constraints

Example: $x_1 + x_2 + x_3 = 15$

$x_1 \geq 0$

$x_2 \geq -2$ $\qquad x_2 = -2 + x_2'$, $\quad x_2' \geq 0$

$x_3 \geq 3$ $\qquad x_3 = 3 + x_3'$, $\quad x_3' \geq 0$

$$x_1 + (-2 + x_2') + (3 + x_3') = 15$$

$$x_1 + x_2' + x_3' = 14 \qquad x_1, x_2', x_3' \geq 0$$

Answer: $\left(\!\binom{3}{14}\!\right) = \binom{3-1+14}{3-1} = \binom{16}{2}$