
I hope to find a contradiction
Lecture 10

# Properties of Implication

| $P$ | $Q$ | $P \Rightarrow Q$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

1) If $P$ is true and $(P \Rightarrow Q)$ is true, then $Q$ is true

   (last row) $(P \land (P \Rightarrow Q)) \Rightarrow Q$ [direct proof]

2) $(\neg Q \Rightarrow \neg P) = (P \Rightarrow Q)$ [Proof by Contrapositive]

3) $(\neg P \Rightarrow \text{False})$ is true, then $P$ is true

   (First row, $\neg P$ is false) [Proof by Contradiction]

4) If $P \Rightarrow Q$ is true, and $(Q \Rightarrow R)$ is true,

   then $(P \Rightarrow R)$ is true. [transitivity]

- All can be verified by truth table

- Let's prove 4) by case analysis on P.
  - If P is false, then $(P \Rightarrow R)$ is true regardless
  - If P is true, then since $(P \Rightarrow Q)$ is true, this make Q true. Now Q is true, and $(Q \Rightarrow R)$ is true. Therefore R is true. Finally $(P \Rightarrow R)$ is true.

# Flash Cards

$P$ true

$P \Rightarrow Q$ is true

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$Q$ is true

$P \Rightarrow Q$ true

$Q \Rightarrow R$ true

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$P \Rightarrow R$ true

$(\neg P \Rightarrow \text{False})$ true

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$P$ true

(contradiction)

$P \Rightarrow Q$ true

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$\neg Q \Rightarrow \neg P$ true

(contrapositive)

We can also say many other things ...

$$\frac{P \Rightarrow Q \text{ true} \qquad P \Rightarrow R \text{ true}}{P \Rightarrow (Q \wedge R) \text{ true}}$$

$$\frac{P \Rightarrow Q \text{ true} \qquad R \Rightarrow W \text{ true}}{(P \wedge R) \Rightarrow (Q \wedge W) \text{ true}}$$

## How to show $(P \Rightarrow Q)$ is true?

- Show that $Q$ is true when $P$ is true (direct)
- Show that $\neg Q \Rightarrow \neg P$ is true (contrapositive)
- Show that $(P \wedge \neg Q) \Rightarrow \text{False}$ is true (contradiction)

## To show $P$ true

- Show that some $R$ is true, and $R \Rightarrow P$ is true (direct)
- Show that $(\neg P \Rightarrow \text{False})$ is true. (contradiction)

Example 1 Show $\overset{P}{n \text{ odd}} \Longrightarrow \overset{Q}{n^2 \text{ odd}}$ (is true)

n odd : $n = 2k+1$ where $k \in \mathbb{Z}$

n even : $n = 2k$ where $k \in \mathbb{Z}$

(Definitions)

$n$ odd $\Rightarrow n = 2k+1 \quad (k \in \mathbb{Z})$

$$n = 2k+1 \Rightarrow n^2 = (2k+1)^2 = 4k^2 + 2k + 1$$

$$= 2(2k^2 + k) + 1 = 2k' + 1 \quad (k' \in \mathbb{Z})$$

$n^2 = 2k'+1 \Rightarrow n^2$ odd

Note: We also established: $n^2$ even $\Longrightarrow$ $n$ even

(contrapositive)

Example 2:

Show : $\overbrace{a \cdot b \notin \mathbb{Q}}^{P} \implies \overbrace{(a \notin \mathbb{Q} \vee b \notin \mathbb{Q})}^{Q}$ (is true)

Consider the contrapositive:

$$\overbrace{a \in \mathbb{Q} \wedge b \in \mathbb{Q}}^{\neg Q} \implies \overbrace{a \cdot b \in \mathbb{Q}}^{\neg P}$$

$Q: A \vee B$

$\neg Q: \neg(A \vee B)$

De Morgan's Law

$\neg A \wedge \neg B$

$a \in \mathbb{Q} \implies a = \dfrac{x}{y}$  $x \in \mathbb{Z},\ y \in \mathbb{N}$

$b \in \mathbb{Q} \implies b = \dfrac{z}{w}$  $z \in \mathbb{Z},\ w \in \mathbb{N}$

$a \in \mathbb{Q} \wedge b \in \mathbb{Q} \implies a = \dfrac{x}{y} \wedge b = \dfrac{z}{w}$

$a = \dfrac{x}{y} \wedge b = \dfrac{z}{w} \implies a \cdot b = \dfrac{x \cdot z}{y \cdot w}$  $xz \in \mathbb{Z},\ yw \in \mathbb{N}$

$a \cdot b = \dfrac{xz}{yw} \implies ab \in \mathbb{Q}.$

## Example 3:

Show $\sqrt{2}$ is irrational ($\sqrt{2} \notin \mathbb{Q}$)

$P: \sqrt{2} \notin \mathbb{Q}$ , $\neg P: \sqrt{2} \in \mathbb{Q}$

$$\underline{\sqrt{2} \in \mathbb{Q}} \Rightarrow \sqrt{2} = \frac{a}{b} \Rightarrow 2 = \frac{a^2}{b^2} \Rightarrow a^2 = 2b^2 \Rightarrow a^2 \text{ even} \Rightarrow \underline{a \text{ even}}$$

from before

$$\underline{\sqrt{2} \in \mathbb{Q}} \Rightarrow b^2 = \frac{a^2}{2} = \frac{(2k)(2k)}{2} = 2k^2 \Rightarrow b^2 \text{ even} \Rightarrow \underline{b \text{ even}}$$

$$\sqrt{2} \in \mathbb{Q} \Rightarrow a \text{ even} \wedge b \text{ even} \Rightarrow \frac{a}{b} \text{ reducible}$$

(cont. next page)

P : $\sqrt{2} \notin \mathbb{Q}$

R : $\dfrac{a}{b}$ irreducible

so :

$$\neg P \Rightarrow \neg R$$

$$\neg P \Rightarrow R \qquad \text{(why?, see next...)}$$

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$$\neg P \Rightarrow \underbrace{(\neg R \wedge R)}_{\text{False}} \qquad \text{Contradiction !}$$

Because:

Every rational number can be expressed as $\frac{a}{b}$ where $a \in \mathbb{Z}$, $b \in \mathbb{N}$, and $\frac{a}{b}$ is irreducible

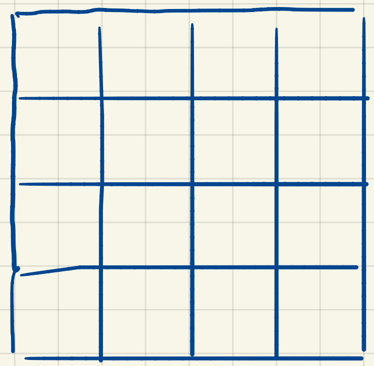
We could have started the proof like this

$$\underline{\sqrt{2} \in \mathbb{Q}} \implies \sqrt{2} = \frac{a}{b} \wedge \underline{\frac{a}{b} \text{ is irreducible}} \implies \ldots$$

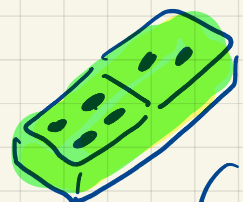
$$\ldots \implies \underline{\frac{a}{b} \text{ is reducible}}$$

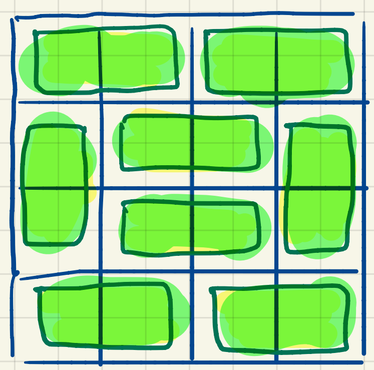
$$\sqrt{2} \in \mathbb{Q} \implies (R \wedge \neg R) \qquad \text{Contradiction.}$$

## Example 4: Consider the $n \times n$ ($n$ even)



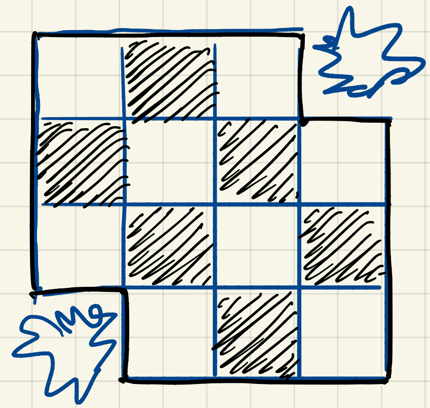
Cover it by dominos.





(all dominos must lie exactly in the grid and occupy two cells)

Show: $\underbrace{\text{opposite corners deleted}}_{P} \Longrightarrow \underbrace{\text{grid not coverable}}_{Q}$



Suppose opposite corners deleted $\wedge$ coverable (is true)

$P \wedge \neg Q$

$P \Rightarrow$ two cells of same color deleted $\Rightarrow$

$\#\text{black} \neq \#\text{white}$

$\neg Q \Rightarrow$ each domino covers exactly one black cell

and one white cell $\Rightarrow$

$\#\text{black} = \#\text{white}$

$$P \wedge \neg Q \Rightarrow \underbrace{(\#b \neq \#w \ \wedge \ \#b = \#w)}_{\text{False}} \quad \text{(contradiction)}$$

so $P \Rightarrow Q$ is true.