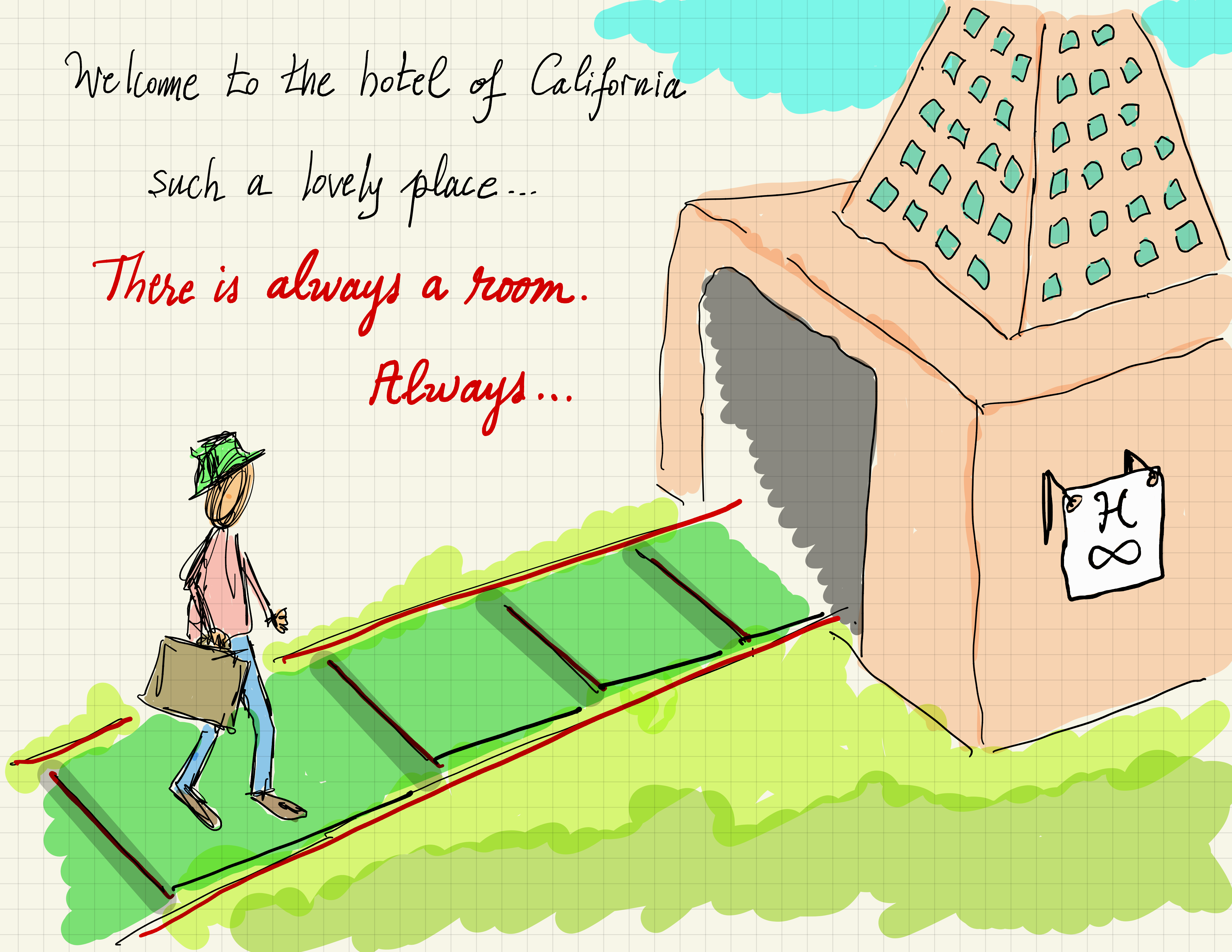

Welcome to the hotel of California
such a lovely place...
There is always a room.
Always...
H
∞

Understanding ∞

Hilbert's Hotel:

- Hilbert has a hotel with infinite number of rooms.
- All rooms are occupied.
- One new customer shows up.
- Hilbert finds a room for the new customer!

How?

Move customer in room i to room $i+1$. Since we have infinitely many rooms, every customer will find a room. Room 1 is now free!

What if infinitely many customers show up? Move customer in room i to room $2i$. Now all odd numbered rooms are free!

So we conclude that $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\{1, 3, 5, \dots\}$ and $\{2, 4, 6, \dots\}$ have the same "size".

Are all infinities the same? **NO**

We define countable sets. A set S is <u>countable</u> if:

- S is finite, or
- $\exists\ f : \mathbb{N} \to S$ and f is a bijection.

  (or $f : S \to \mathbb{N}$)

Consider $f: \mathbb{N} \to \{2, 4, 6, \ldots\}$

$$f(x) = 2x$$

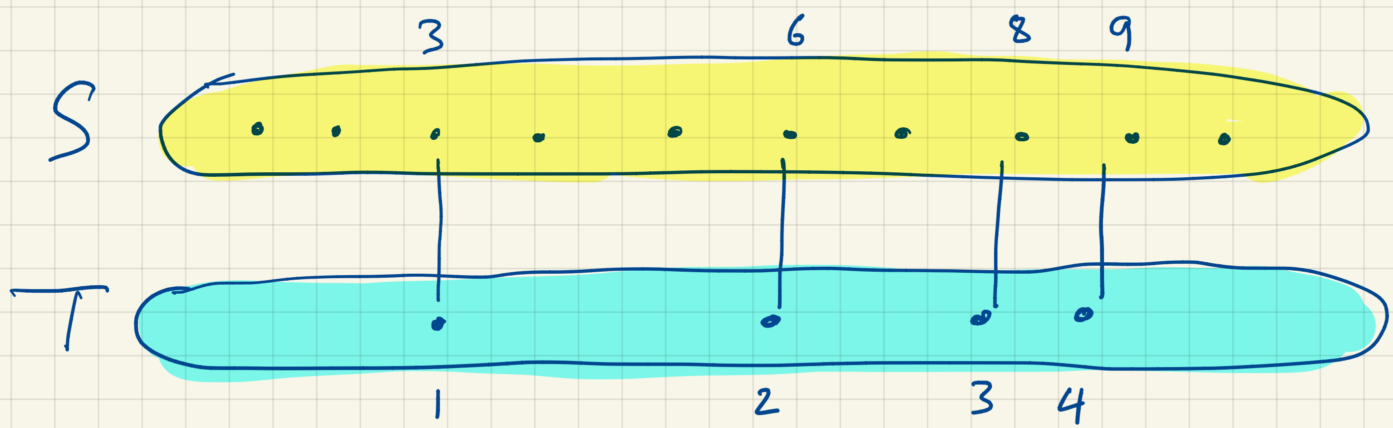
f is one-to-one: $x_1 \neq x_2 \Rightarrow 2x_1 \neq 2x_2 \Rightarrow f(x_1) \neq f(x_2)$

f is onto: $\forall\, y \in \{2, 4, 6, \ldots\}$. $x = \frac{y}{2} \in \mathbb{N}$ and $f(x) = y$

In other words: A set S is countable if we can give a specific order on its elements such that each element has a <u>finite</u> rank in that order.

This suggests the following results:

- If S is countable, then $T \subset S$ is countable
  - preserve relative order of elements in T, each will have a rank that is at most it's rank in S, so finite



- If A and B are countable, the $A \cap B$ is countable.

  $A \cap B \subset A$.

- If A and B are countable, then $A \cup B$ is countable.

$$A = \{a_1, a_2, a_3, \ldots\} \qquad B = \{b_1, b_2, b_3, \ldots\}$$

Can we list elements in $A \cup B$ such that each will have a finite rank?

$$a_1 \quad a_2 \quad a_3 \quad \ldots \quad b_1 \quad b_2 \quad b_3 \qquad ✗$$

$$a_1 \quad b_1 \quad a_2 \quad b_2 \quad a_3 \quad b_3 \quad \ldots \qquad ✓$$

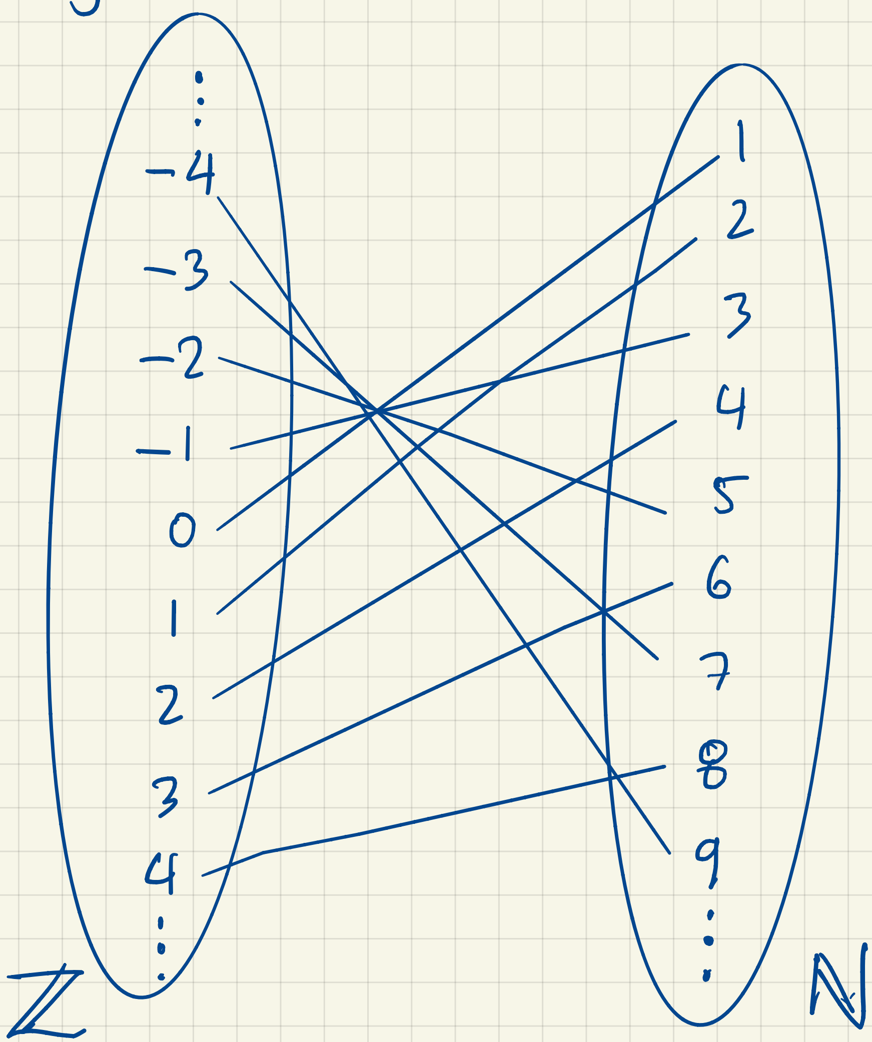
Each element at most doubles its rank, so finite.

Is $\mathbb{Z}$ countable?

$0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots$

If $i \in \mathbb{Z}$ is positive, it's rank is $2i$, so finite rank.

If $i \in \mathbb{Z} \leqslant 0$, it's rank is $2|i|+1$, so finite rank.



$$f(x) = \begin{cases} 2x & x > 0 \\ -2x+1 & x \leq 0 \end{cases}$$

Bijection

Is $\mathbb{Q}$ countable?

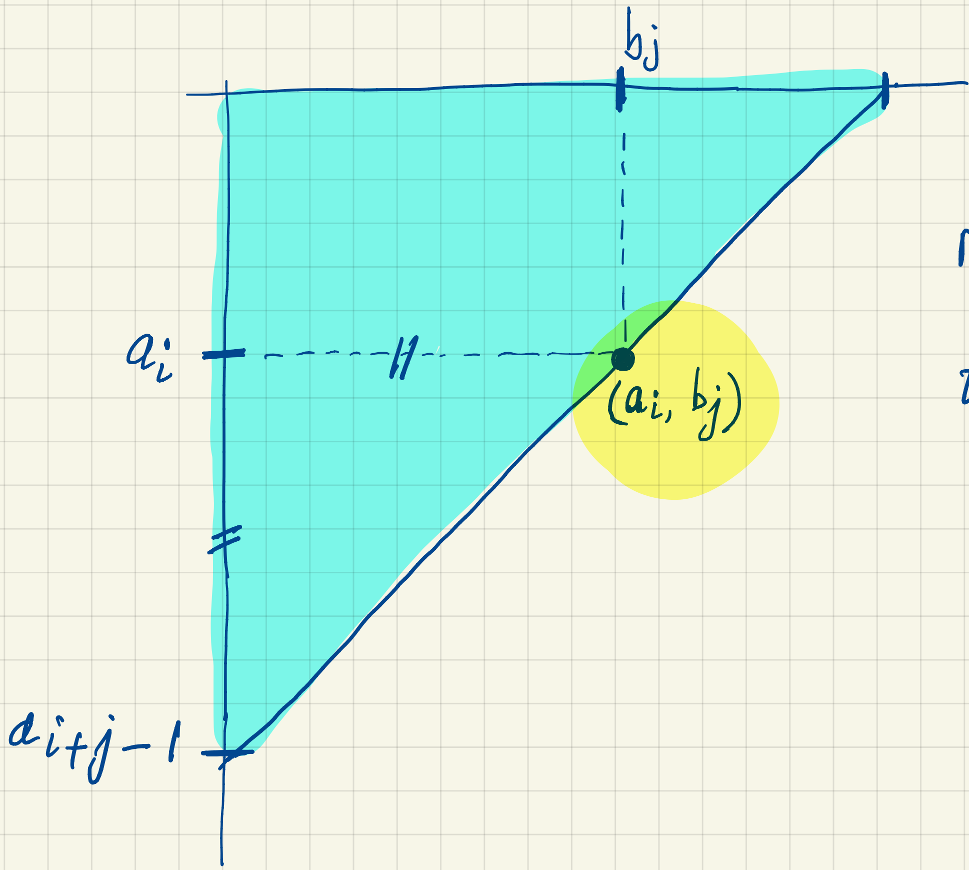
First if A and B are countable, then $A \times B$ is countable.

$A = \{a_1, a_2, a_3, \dots\}$ $\quad B = \{b_1, b_2, b_3, \dots\}$

	b_1	b_2	b_3	b_4	$\dots$
a_1	(a_1, b_1)	(a_1, b_2)	(a_1, b_3)	(a_1, b_4)	
a_2	(a_2, b_1)	(a_2, b_2)	(a_2, b_3)	(a_2, b_4)	
a_3	(a_3, b_1)	(a_3, b_2)	(a_3, b_3)	(a_3, b_4)	
a_4	(a_4, b_1)	(a_4, b_2)	(a_4, b_3)	(a_4, b_4)	
$\vdots$					

This "diagonal" order guarantees finite rank for every (a_i, b_j)

Proof :



rank of $(a_i, b_j) \leq$ # elements in triangle $= 1+2+3+\cdots+\underbrace{(i+j-1)}_{n}$

$$= \frac{\overbrace{(i+j-1)}^{n}\overbrace{(i+j)}^{n+1}}{2} \quad (\text{finite}).$$

$\mathbb{Q}$ can be thought of as a "subset" of $\mathbb{Z} \times \mathbb{N}$.
So $\mathbb{Q}$ is countable! Every fraction $\frac{a}{b}$ is a "pair" where $a \in \mathbb{Z}$ and $b \in \mathbb{N}$.

$\mathbb{R}$ is NOT countable (Diagonalization)

There is no bijection from $\mathbb{N}$ to $\mathbb{R}$.

Proof by contradiction: Suppose there is such bijection, $f: \mathbb{N} \to \mathbb{R}$
then for every $i \in \mathbb{N}$, let $f(i) = r \in \mathbb{R}$.

Construct $x \in \mathbb{R}$ such that the i^{th} digit of x
is different than the i^{th} digit of $f(i)$

e.g.:

$\mathbb{N}$	$\mathbb{R}$
1	0.123
2	3.14
3	5.555
4	2.5000
$\vdots$	

$x = 0.\bar{1}\bar{4}\bar{5}\bar{0}\ldots$

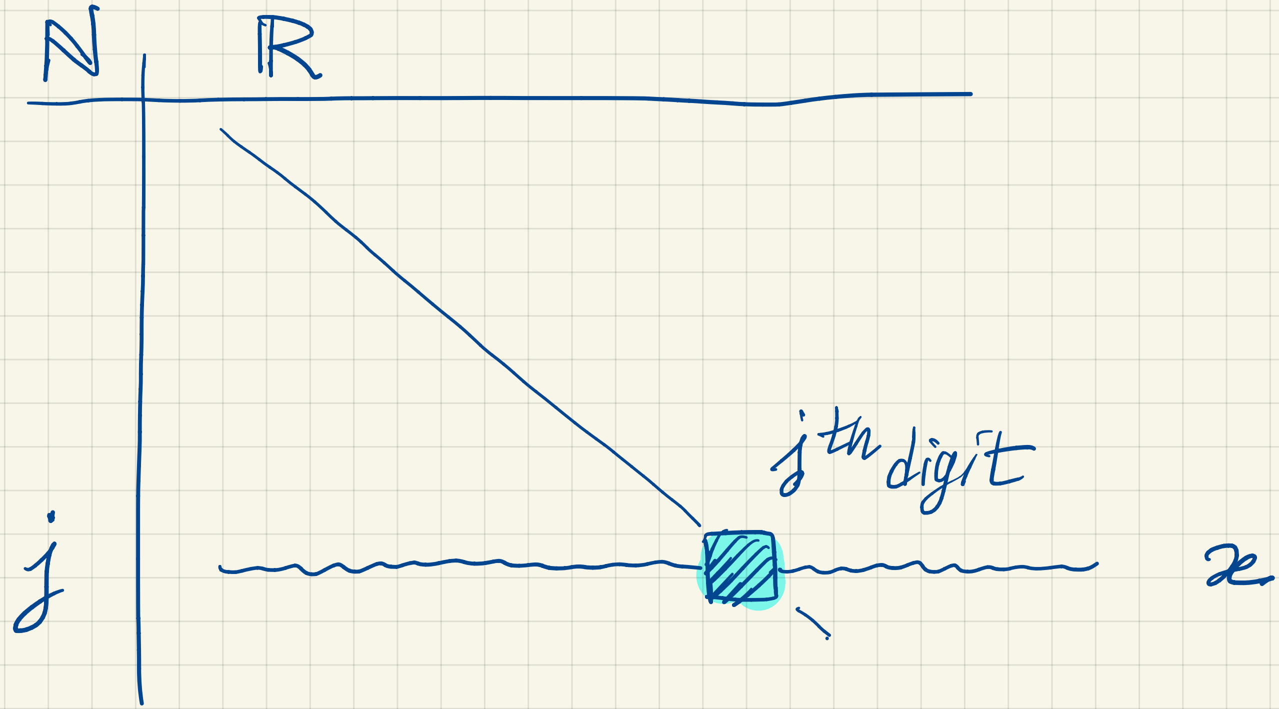
Since $x \in \mathbb{R}$ and

$\forall j \in \mathbb{N}.\ f(j) \neq x$

then f is not onto.

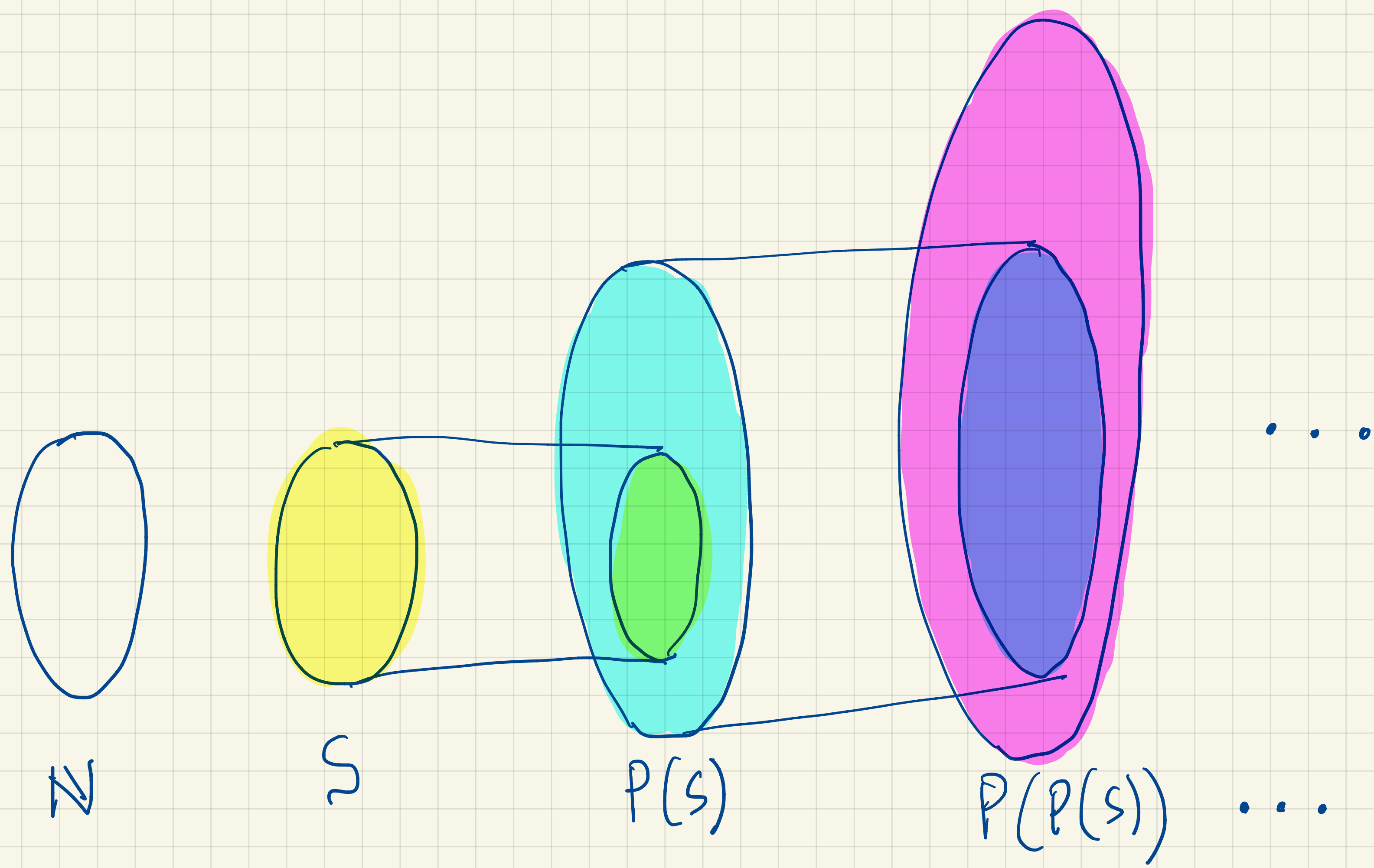
Contradiction.

Another way: Since f is bijection and $x \in \mathbb{R}$, then $\exists\ j$ such that $f(j) = x$. But then j^{th} digit of x is different than j^{th} digit of x. Contradiction.



Power sets.

- We can show that there is NO bijection from S to $\mathcal{P}(S)$. (similar diagonalization method)
- There are infinite levels of infinity. ($\mathbb{N}$ smallest).
- This means the power set of any infinite set is not countable.



$\mathbb{N}$
S
$P(S)$
$P(P(S))$
...