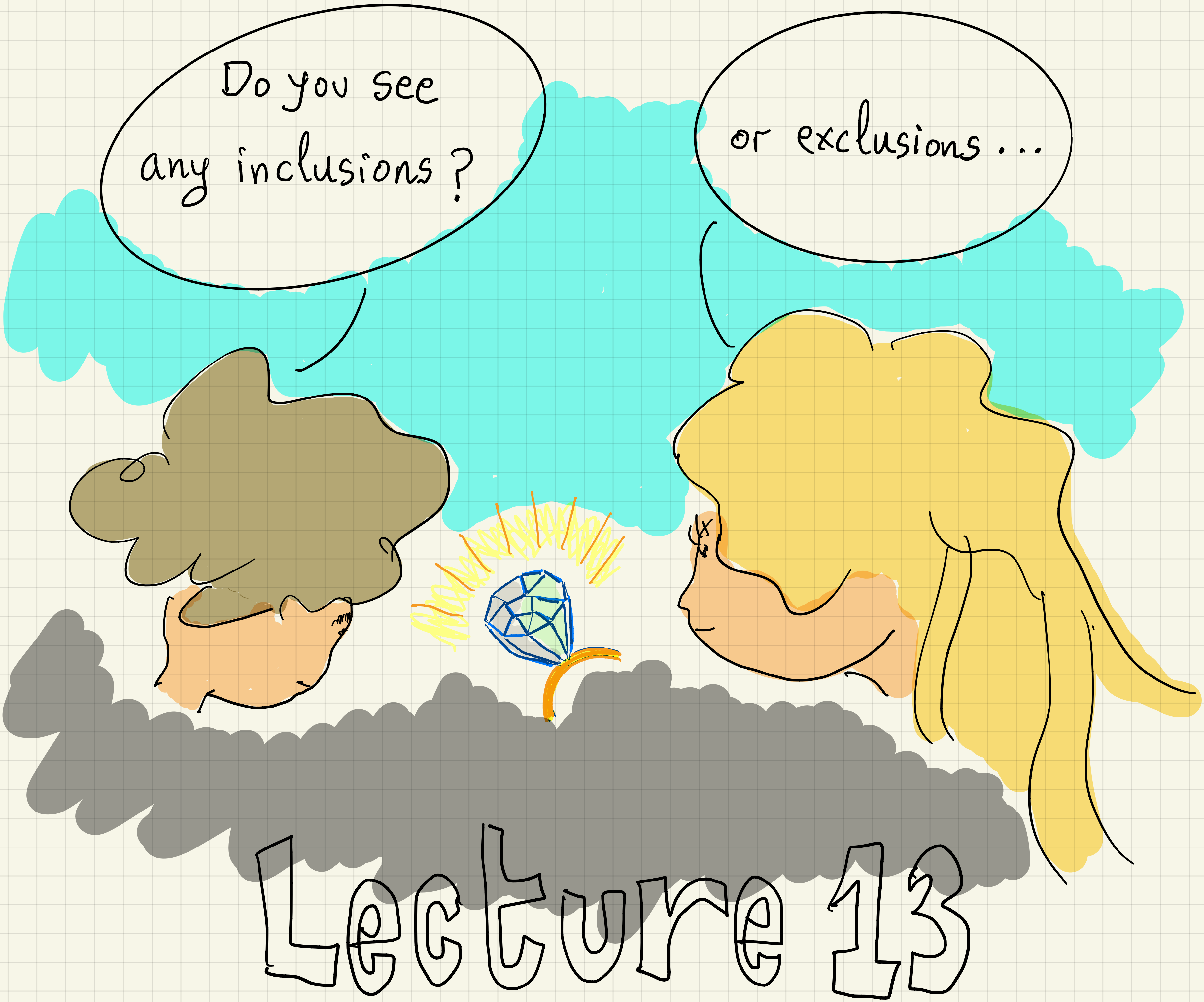
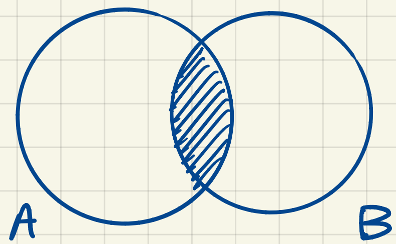

Do you see any inclusions?
or exclusions...
Lecture 13

The Inclusion-Exclusion Principle

Most basic form with two sets



$$|A \cup B| = |A| + |B| - |A \cap B|$$

(why?)

Example 1 : I have 50 socks, 35 are black and 30 are cotton. How many black cotton socks do I have?

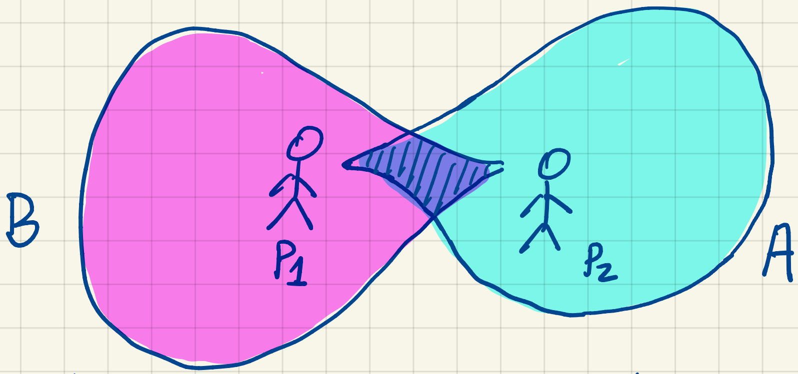
$A = \{x : x \text{ is black sock}\}$ $\quad B = \{x : x \text{ is cotton sock}\}$

$$|A \cup B| = |A| + |B| - |A \cap B| \Rightarrow 50 = 35 + 30 - |A \cap B|$$

$$|A \cap B| = 15.$$

Example 2 : A bit more involved.

In a group of 12 people, each person knows more than 6 other people. Show that we can find 3 people that know each other.



people known by P_2

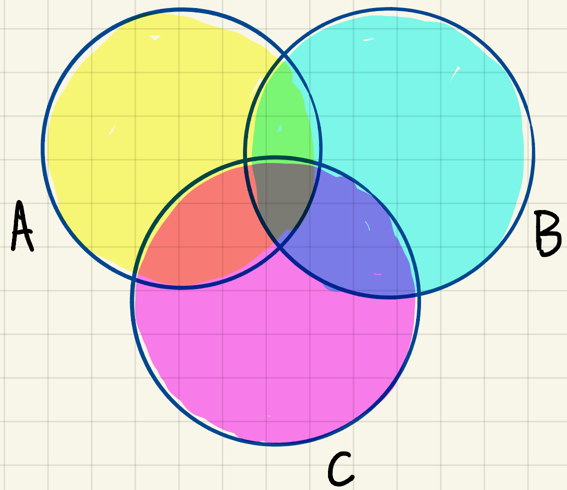
people known by P_1

$$|A \cap B| = \underset{>6}{|A|} + \underset{>6}{|B|} - \underset{\leq 12}{|A \cup B|}$$

$$|A \cap B| > 6 + 6 - 12 = 0$$

$$|A \cap B| > 0 \quad \text{(Done)}$$

What about more sets?

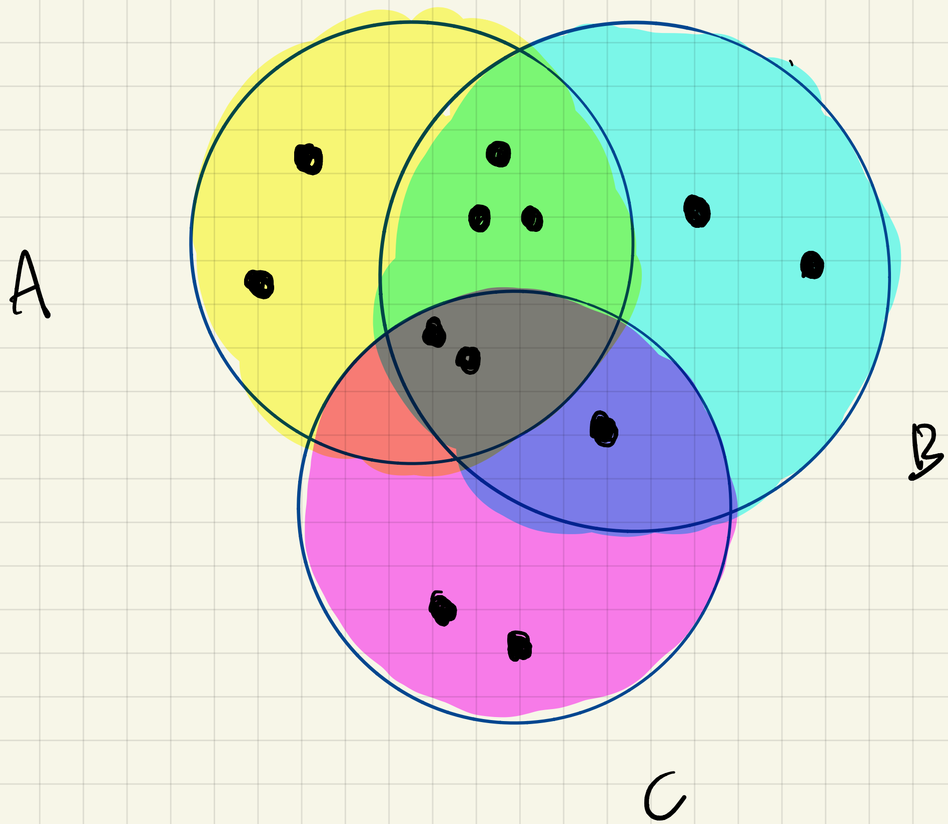


$$|A \cup B \cup C| = |A| + |B| + |C| \quad \text{(include)}$$
$$- |A \cap B| - |A \cap C| - |B \cap C| \quad \text{(exclude)}$$
$$+ |A \cap B \cap C| \quad \text{(include)}$$

<u>Four sets:</u>

<u># terms</u>

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| \quad \text{(include)} \quad \binom{4}{1}$$
$$- |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \quad \text{(Exclude)} \quad \binom{4}{2}$$
$$+ |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \quad \text{(include)} \quad \binom{4}{3}$$
$$- |A \cap B \cap C \cap D| \quad \text{(Exclude)} \quad \binom{4}{4}$$



$|A| = 7$

$|B| = 8$

$|C| = 5$

$|A \cap B| = 5$

$|A \cap C| = 2$

$|B \cap C| = 3$

$|A \cap B \cap C| = 2$

$$\text{Total} = 7 + 8 + 5$$
$$- 5 - 2 - 3$$
$$+ 2 = 12$$

<u>Rule</u>: Include all singletons, Exclude all pairs, Include all triplets, ...

Why Does it work?

Consider a given element and suppose it belongs to $n > 0$ sets

It belongs to $\binom{n}{2}$ pairs of sets

and $\binom{n}{3}$ triplets of sets

and $\binom{n}{4}$ quadruplets of sets

$\vdots$

The Inclusion - Exclusion principle for this element contributes

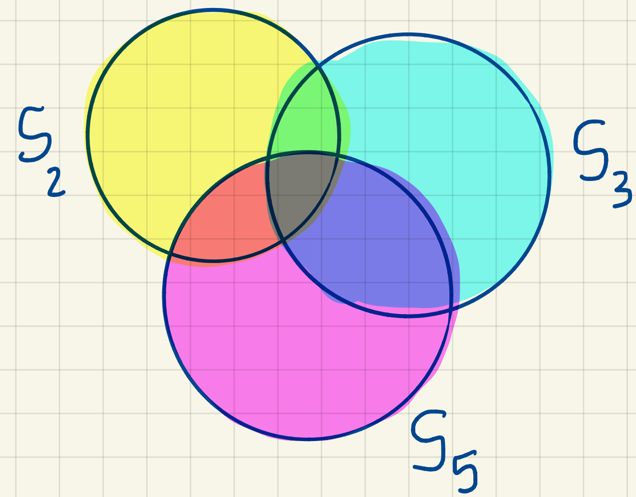
$$\binom{n}{1} - \binom{n}{2} + \binom{n}{3} - \binom{n}{4} + \cdots \binom{n}{n} =$$

$$\binom{n}{0} - \left[\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots \binom{n}{n}\right] = 1 - 0 = 1.$$

So each element contributes 1.

Example 3:

How many positive integers ≤ 1000 are divisible by 2 or 3 or 5.



Recall: Union is OR logic.

$S_i = \#$ elem divisible by i

We want:

$$|S_2 \cup S_3 \cup S_5| = |S_2| + |S_3| + |S_5| - |S_2 \cap S_3| - |S_2 \cap S_5| - |S_3 \cap S_5| + |S_2 \cap S_3 \cap S_5|$$

$$|S_2| = \frac{1000}{2} = 500 \qquad |S_3| = \left\lfloor \frac{1000}{3} \right\rfloor = 333 \qquad |S_5| = \frac{1000}{5} = 200$$

$$|S_2 \cap S_3| = \left\lfloor \frac{1000}{6} \right\rfloor = 166 \qquad |S_2 \cap S_5| = \frac{1000}{10} = 100 \qquad |S_3 \cap S_5| = \left\lfloor \frac{1000}{15} \right\rfloor = 66$$

$$|S_2 \cap S_3 \cap S_5| = \left\lfloor \frac{1000}{30} \right\rfloor = 33$$

Finally $|S_2 \cup S_3 \cup S_5| = 500 + 333 + 200 - 166 - 100 - 66 + 33 = ?$

<u>Example 4</u> : Good Passwords. How many n char. passwords can we make if the password must contain at least one upper case letter, one lower case letter, and one digit ?

This is an <u>AND</u> logic. Product rule will be hard to use due to overcounting. One might be tempted to say

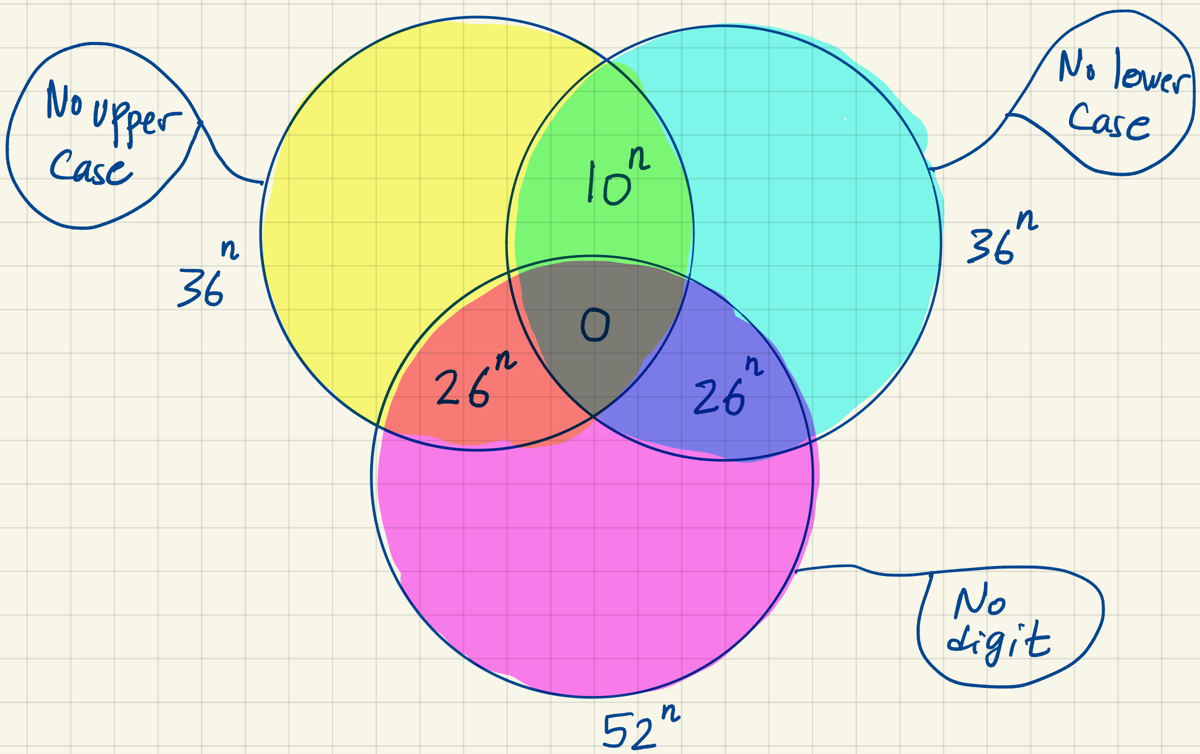
- Pick 3 char of the n char. with order.
- Make them upper case letter, lower case letter, digit
- Choose the remaining $n-3$ char from 62 possibilities

Giving $\dfrac{n!}{(n-3)!} \times 26 \times 26 \times 10 \times 62^{n-3}$

But this overcounts and it's hard to deal with because the overcount depends on the choices of the $n-3$ remaining letters.

Transform to OR logic by negating. So count **Bad** passwords.

How many passwords don't have an upper case, or a lower case letter, or a digit?



$$\# \text{Good passwords} = 62^n - \left(36^n + 36^n + 52^n - 26^n - 26^n - 10^n + 0\right)$$

The Lazy Professor (revisited)

How many permutations assign NO student to his/her own test?

Convert to OR logic: Count bad permutations.

Let S_i = set of bad permutations for student i.

(S_i assigns student i his/her test)

We want # bad perm. $= |S_1 \cup S_2 \cup \ldots \cup S_n|$

		# terms
# permutations that assign student i to his test.	$(n-1)!$	n
# " " " students (i,j) to their test	$(n-2)!$	$\binom{n}{2}$
# " " " students (i,j,k) to their test	$(n-3)!$	$\binom{n}{3}$

$\vdots$

$\#$ bad permutations $= (n-1)!\binom{n}{1} - (n-2)!\binom{n}{2} + (n-3)!\binom{n}{3} - \cdots\cdots$

$\#$ Good permutations $=$

$$n! - (n-1)!\binom{n}{1} + (n-2)!\binom{n}{2} - (n-3)!\binom{n}{3} + \cdots\cdots$$

$$= \frac{n!}{0!} - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \cdots + (-1)^n \frac{n!}{n!}$$

$$= n! \underbrace{\left[\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!} \right]}_{1/e \text{ for large } n}$$

$$\approx \frac{n!}{e}$$

(prob. of good permutation is $\frac{1}{e} \approx 0.3678$)