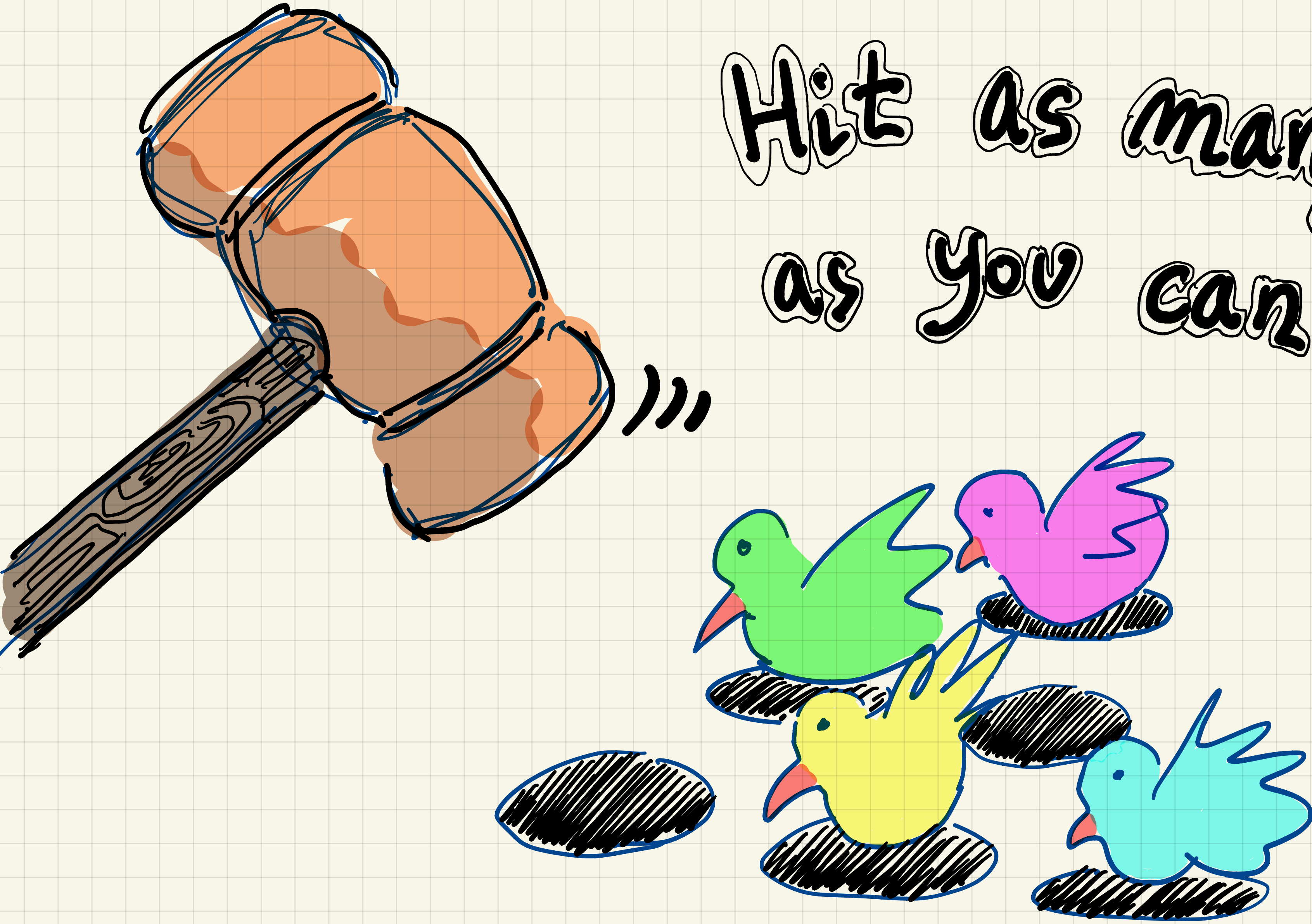


Pigeonhole, not what you might think

The Pigeonhole Principle

I will explain what Pigeonhole is, but for now consider the following problems

* Choose 51 numbers in $\{1, 2, \dots, 100\}$. Prove that 2 of them must be consecutive $(x, x+1)$

* Place 10 points inside 3×3 square. Prove that 2 of them must be within a distance of $\sqrt{2}$.

* Place the numbers $1, 2, \dots, 10$ in 3 bins. Show that the sum in one bin must be at least 19.

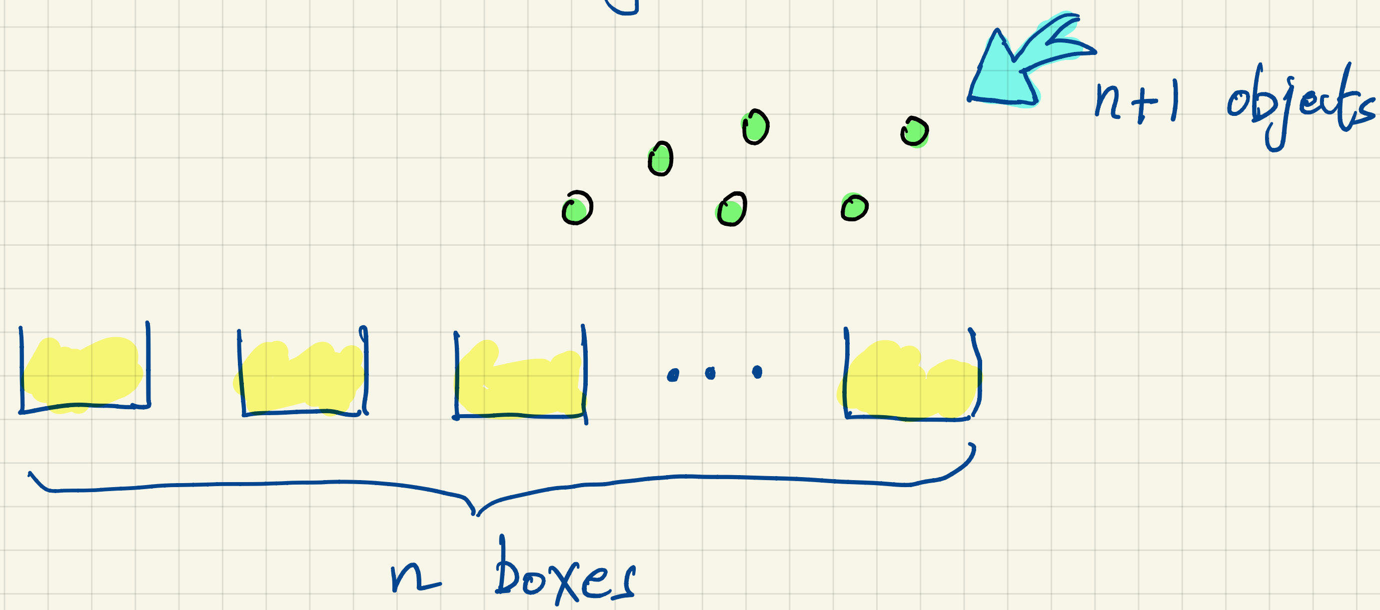
What's common among the above problems?

- First, they give you freedom in doing something in any way you want.
- Then, they claim something will always happen no matter how you do it.

When you sense this, think Pigeonhole!

Pigeonhole: Basic form

Given n boxes and $n+1$ objects, when we place all objects in boxes, at least one box will contain at least 2 objects.



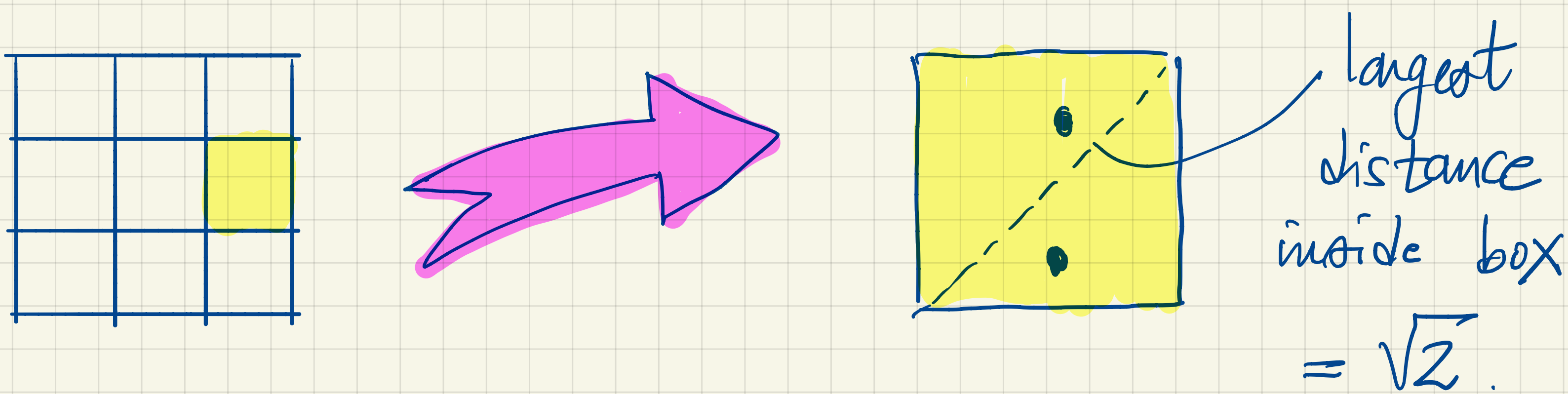
Proof by contradiction: Let x_i = # objects in box i and suppose $\forall i.\ x_i \leq 1$. Then $\sum_{i=1}^{n} x_i \leq \underbrace{1+1+\cdots+1}_{} = n$.

Contradiction!

Applications:

Given a 3x3 square, place 10 points in it. Prove that 2 of them are within a distance of $\sqrt{2}$.

- First, think pigeonhole
- Second, set it up: where are the objects & boxes
  - If 10 points are objects, we need 9 boxes
  - where do we see nine?



Choose 51 numbers in $\{1, 2, \ldots, 100\}$. Prove 2 of them must be consecutive?

- Sounds like Pigeonhole.
- Need 50 boxes?

$$\boxed{1,2} \quad \boxed{3,4} \quad \boxed{5,6} \quad \cdots \quad \boxed{99,100}$$

Choosing a number $\Longrightarrow$ place a token in its box.
51 tokens, 50 boxes $\Longrightarrow$ 1 box will contain 2 tokens. So two numbers will be consecutive.

Generalization

Given n boxes and m objects placed in the boxes, at least one box will contain at least $\lfloor \frac{m-1}{n} \rfloor + 1$ objects.

Proof: let $x_i = \#$ objects in box i and suppose

$\forall i.\ x_i \leq \lfloor \frac{m-1}{n} \rfloor$. Then

$$\sum_{i=1}^{n} x_i \leq \lfloor \frac{m-1}{n} \rfloor + \lfloor \frac{m-1}{n} \rfloor + \cdots + \lfloor \frac{m-1}{n} \rfloor = n \lfloor \frac{m-1}{n} \rfloor$$

$$\leq n \frac{m-1}{n} = m-1.$$ Contradiction.

: If m, n positive integers, then

$$\lfloor \frac{m-1}{n} \rfloor + 1 = \lceil \frac{m}{n} \rceil$$

some box must contain at least the average

Application: Place the numbers 1, 2, ..., 10 in 3 bins.

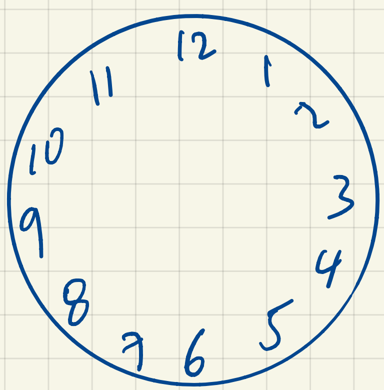
- Prove at least one bin contains at least 4 numbers.

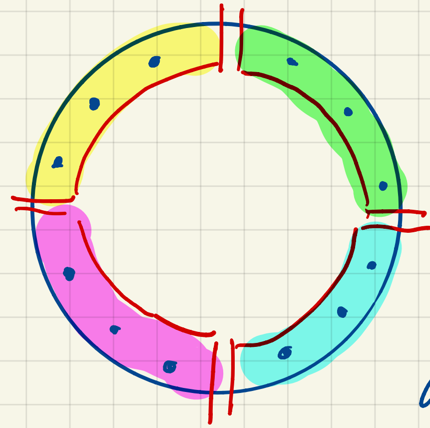
  Pigeonhole: $n=3$, $m=10$. $\left\lceil \frac{m}{n} \right\rceil = \left\lceil \frac{10}{3} \right\rceil = 4$

- Prove at least one bin contains a sum of at least 19.

  Pigeonhole: $n=3$, $m = 1+2+\cdots+10 = 55$. $\left\lceil \frac{55}{3} \right\rceil = 19$

Scrambled Clock: Scramble the numbers. Show there are 3 consecutive numbers that add up to at least 20





"Divide" the clock into boxes of 3.

One box must contain a sum of at least

$$\left\lceil (1+2+\cdots+12)/4 \right\rceil = 20.$$

- Sometimes, we have to be "smart" about how to set up the problem.

- Here's an example: The numbers 1 to 10 are written down in some order. Show there are 3 consecutive numbers that add up to at least 15.

• • • | • • • | • • • | •

One of the 4 boxes must contain a sum of at least

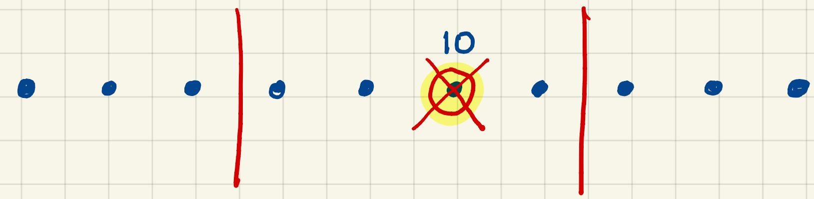
$$\left\lceil \frac{55}{4} \right\rceil = 14.$$

Does not give us what we want!!!

Idea: 4 boxes too many.

Remove largest number, 10.

<u>example:</u>



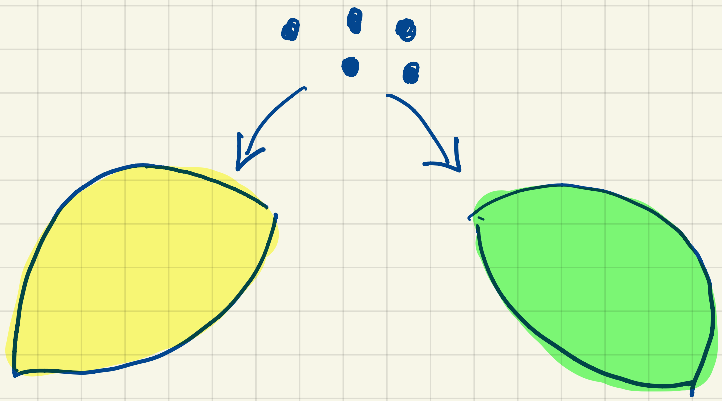
One box must contain a sum of at least $\left\lceil \frac{45}{3} \right\rceil = 15$.

But what if it's the second box?

- Not consecutive any more
- Replace last number in box by 10
- Since 10 is largest, still works

Another example of being "smart" in setting up the pigeonhole.

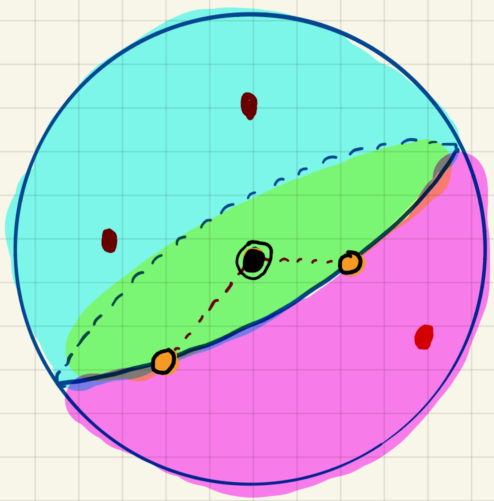
Place 5 points on surface of sphere. Prove that 4 of them lie in the same hemisphere.



By Pigeonhole one hemisphere must contain $\lceil \frac{5}{2} \rceil = 3$ points.

Does not give us what we want!

Idea: Choose the hemispheres.



Choose hemispheres defined by the center and 2 points, as shown.

Pigeonhole: Among the 3 remaining points, one hemisphere will contain at least $\lceil \frac{3}{2} \rceil = 2$ points.

Therefore, this hemisphere contains 4 points. Done.

An alternative way to look at Pigeonhole

Given n numbers $x_1, x_2, \ldots, x_n$ such that

$$x_1 + x_2 + \ldots + x_n = m$$

Then $\exists x_i .\ x_i \geq \frac{m}{n}$

("One number must be at least the average")

x_i : # objects in box i

n : # boxes

m : # objects