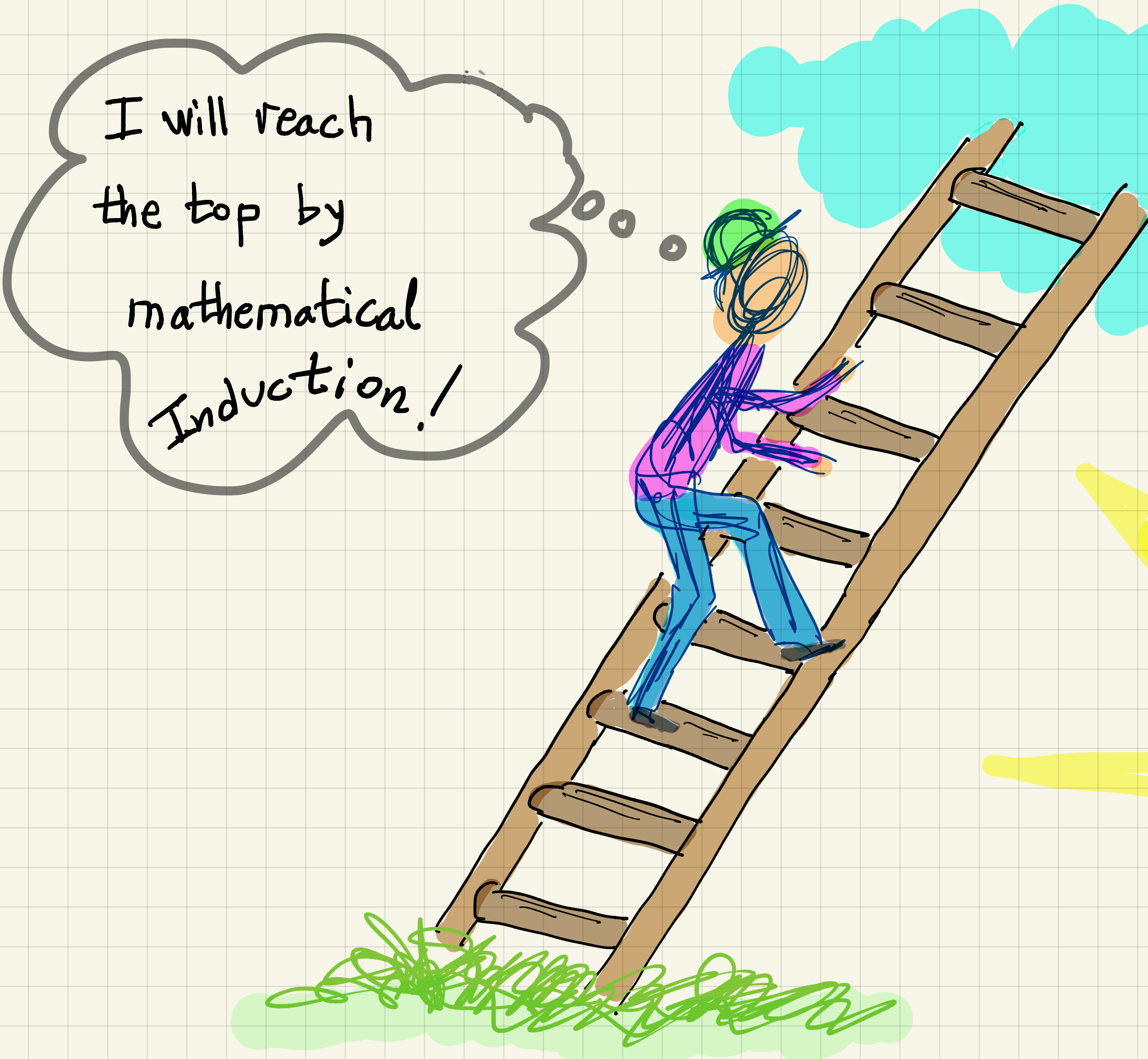

lecture 15

If I can get on the bottom level (Base case) and if on the assumption (Inductive hypothesis) that I have reached some arbitrary level, I can (Inductive step) climb to the next level; then I can (Conclusion) reach any level.

Mathematical Induction

- Proof technique
- Proving some properties of integers.

  e.g. $\forall n \in \mathbb{N}.\ P(n)$

- In many cases we take $\mathbb{N} = \{0, 1, 2, 3, \dots\}$

Examples:

- Prove that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.
- Prove that every positive integer n can be written as $n = m2^k$ where m is odd
- Prove that $\underbrace{1 + 3 + 5 + \dots + (2n-1)}_{\text{first } n \text{ odd numbers}} = n^2$ for all $n \geq 1$

How to prove by induction?

To Prove $P(n)$ is true $\forall\, n \geq n_0$ (Typically $n_0 = 0$)

- Base case : Prove $P(n_0)$ is true
- Inductive step: Prove $\forall\, k \geq n_0.\ P(k) \Rightarrow P(k+1)$

  In other words, assume $P(k)$ is true (inductive hypothesis) and prove $P(k+1)$ is also true.

That's it!

<u>Example 1:</u> Prove that $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ for all $n \geq 0$

<u>Base case:</u> $P(0)$: $\sum_{i=0}^{0} 2^i = 2^{0+1} - 1$

$$2^0 = 1 \checkmark$$

<u>Inductive step:</u> $\forall k \geq 0.\ P(k) \Rightarrow P(k+1)$

$P(k)$: $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$ (Inductive hypothesis)

$P(k+1)$: $\sum_{i=0}^{k+1} 2^i = 2^{k+2} - 1$

$$\sum_{i=0}^{k+1} 2^i = \sum_{i=0}^{k} 2^i + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$$

$$= 2 \cdot 2^{k+1} - 1 = 2^{k+2} - 1 .$$ <u>Done</u>

Example 2: Prove $\prod_{i=1}^{n}\left(1+\frac{1}{i}\right) = n+1$ for all $n \geq 1$

Base case: $n_0 = 1$. $P(1)$: $\prod_{i=1}^{1}\left(1+\frac{1}{i}\right) = 1+1$

$1+\frac{1}{1} = 2$ ✓

Inductive step $\forall k \geq 1.\ P(k) \Rightarrow P(k+1)$

$P(k)$: $\prod_{i=1}^{k}\left(1+\frac{1}{i}\right) = k+1$ (Inductive hypothesis)

$P(k+1)$: $\prod_{i=1}^{k+1}\left(1+\frac{1}{i}\right) = (k+1)+1$

$$\prod_{i=1}^{k+1}\left(1+\frac{1}{i}\right) = \left(\prod_{i=1}^{k}\left(1+\frac{1}{i}\right)\right)\left(1+\frac{1}{k+1}\right) = (k+1)\left(1+\frac{1}{k+1}\right)$$

$$= k+1+1 = k+2$$ ✓

Example 3.

$$T_0 = 0$$
$$T_{n+1} = T_n \overline{T_n}$$

$T_1 = T_0 \overline{T_0} = 01$

$T_2 = T_1 \overline{T_1} = 0110$

$T_3 = T_2 \overline{T_2} = 01101001$

$\vdots$

Prove T_n has 2^n bits $\forall n \in \mathbb{N}$

Base case: $P(0)$. T_0 has 1 bit ✓

Inductive step: $\forall k \geq 0.\ P(k) \Rightarrow P(k+1)$

$P(k)$: T_k has 2^k bits (Inductive hypothesis)

$P(k+1)$: T_{k+1} has 2^{k+1} bits

$T_{k+1} = T_k \overline{T_k}$ which has $2 \times 2^k = 2^{k+1}$ bit. Done.

Example 4. Prove that T_n starts with 01 for $n \geq 1$.

Base case: $P(1)$: T_1 starts with 01. (True) ✓

Inductive step: $\forall k \geq 1.\ P(k) \Rightarrow P(k+1)$

$P(k)$: T_k starts with 01 (Inductive step)

$P(k+1)$: T_{k+1} starts with 01

$T_{k+1} = T_k \overline{T_k}$. T_{k+1} starts with T_k, so it starts with 01.

Prove that T_n ends in 10 if n is even and in 01 if n is odd, for all $n \geq 1$.

Base case: $P(1)$: T_1 ends in 01. True ✓

Inductive step: $\forall\, k \geq 1.\ P(k) \Rightarrow P(k+1)$

$P(k)$: T_k ends in 10 if k is even and in 01 if k is odd

$T_{k+1} = T_k \overline{T_k}$

- $k+1$ odd $\Rightarrow$ k even $\Rightarrow$ T_k ends in 10 $\Rightarrow$ $\overline{T_k}$ ends in 01 $\Rightarrow$ T_{k+1} ends in 01
- $k+1$ even $\Rightarrow$ you do it.

Prove that $n^3 - n$ is a multiple of 3 $\forall\, n \in \mathbb{N}$

Base case: $n_0 = 0$. $P(0)$: $0^3 - 0 = 3m$ True ✓

Inductive step: $\forall\, k \geqslant 0.\ P(k) \Rightarrow P(k+1)$

$P(k)$: $k^3 - k = 3m$ (Inductive hypothesis)

$P(k+1)$: $(k+1)^3 - (k+1) = 3m'$

$$
\begin{aligned}
(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\
&= (k^3 - k) + 3(k^2 + k) \\
&= 3m + 3(k^2 + k) \\
&= 3(m + k^2 + k) = 3m'.
\end{aligned}
$$

What's wrong with this:

Prove $\sum_{i=1}^{n} i = \frac{n^2+n+\sqrt{\pi}}{2}$, $\forall n \in \mathbb{N}$

$P(k)$: $\sum_{i=1}^{k} i = \frac{k^2+k+\sqrt{\pi}}{2}$ (Inductive hypothesis)

$P(k+1)$: $\sum_{i=1}^{k+1} i = \frac{(k+1)^2+(k+1)+\sqrt{\pi}}{2}$

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1) = \frac{k^2+k+\sqrt{\pi}}{2} + \frac{2(k+1)}{2} = \frac{(k+1)^2+(k+1)+\sqrt{\pi}}{2}$$

??? :(

$\forall n \geq 2$. n lines no two of which are $//$ intersect in one point.

<u>Base case:</u> $n_0 = 2$. $P(2)$: 2 lines not $//$ intersect in one point ✓

<u>Inductive step:</u> $\forall k \geq 2$. $P(k) \Longrightarrow P(k+1)$

Given $k+1$ lines $\ell_1, \ell_2, \ell_3, \dots, \ell_k, \ell_{k+1}$ no two of which are $//$,

Consider the two sets of lines

- $\ell_1, \ell_2, \ell_3, \dots, \ell_{k-1}, \ell_{k+1}$ (no two are $//$)
- $\ell_1, \ell_2, \ell_3, \dots, \ell_{k-1}, \ell_k$ (no two are $//$)

Each set has k lines $\Longrightarrow$ all lines in each set intersect in one point, namely the intersection of ℓ_1 & ℓ_2.

Therefore all $k+1$ lines go through that point !!!!

Here's another of those :

For every $n \geqslant 12$, $n = 3x + 7y$ where $x, y \in \mathbb{N}$.

Base case: $n_0 = 12$. $P(12)$: $12 = 3 \times 4 + 7 \times 0$ ✓

Inductive step: $\forall k \geqslant 12.\ P(k) \Rightarrow P(k+1)$

$$P(k) : k = 3x + 7y \ , \quad x, y \in \mathbb{N}$$

$$P(k+1) : k+1 = 3x' + 7y' \ , \quad x', y' \in \mathbb{N}$$

$$k+1 = 3x + 7y + 1 = 3x + 7y + \underbrace{7 - 2 \times 3}$$

$$= 3(x-2) + 7(y+1)$$

$\in \mathbb{N}$?

Recitation: (simple induction)

- Prove $$\sum_{i=1}^{n} (2i-1) = n^2 \quad \text{for all } n \geq 0$$

- Prove $$\sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1} \quad \text{for all } n \geq 0 \quad (a \neq 1)$$

- Prove that a $2^n \times 2^n$ chessboard can be covered by L-shaped triminos if one squared is removed, for all $n \geq 1$.

Quick overview of inductive steps for above.

- $$\sum_{i=1}^{k+1}(2i-1) = \sum_{i=1}^{k}(2i-1) + 2(k+1)-1 = k^2 + 2(k+1)-1$$
$$= k^2 + 2k + 1 = (k+1)^2$$

- $$\sum_{i=0}^{k+1} a^i = \sum_{i=0}^{k} a^i + a^{k+1} = \frac{a^{k+1}-1}{a-1} + a^{k+1}$$
$$= \frac{a^{k+1}-1+a^{k+2}-a^{k+1}}{a-1} = \frac{a^{k+2}-1}{a-1}$$

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