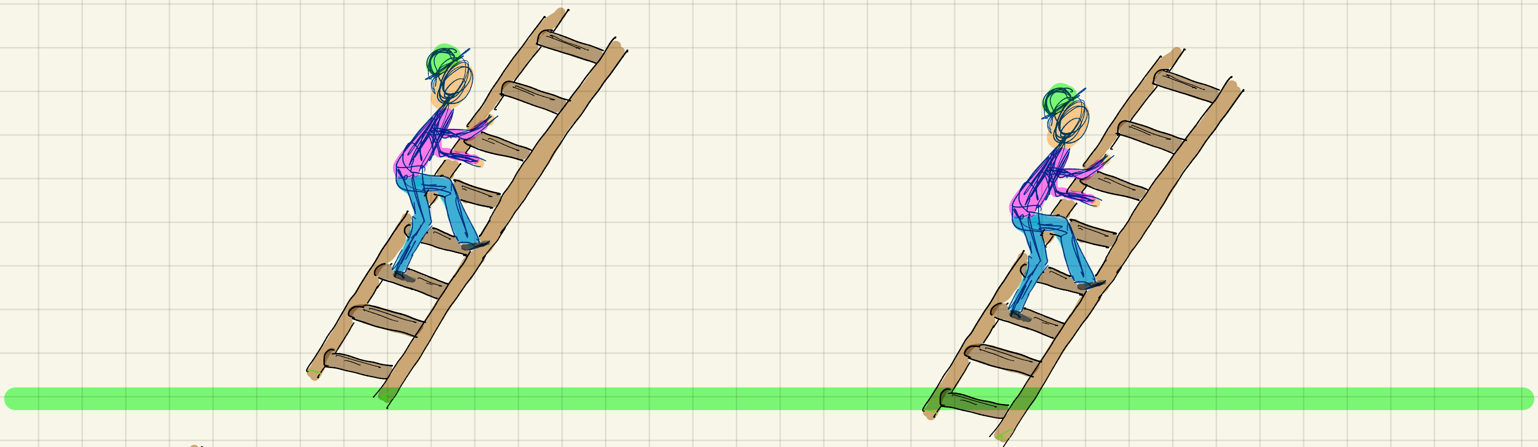
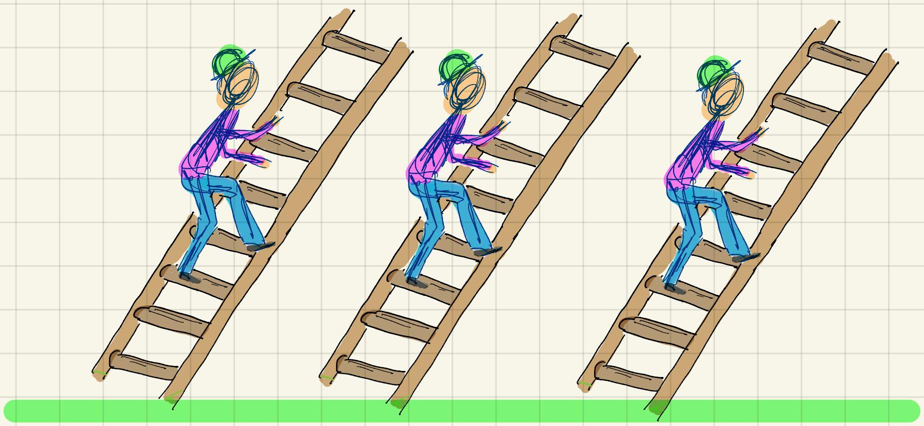
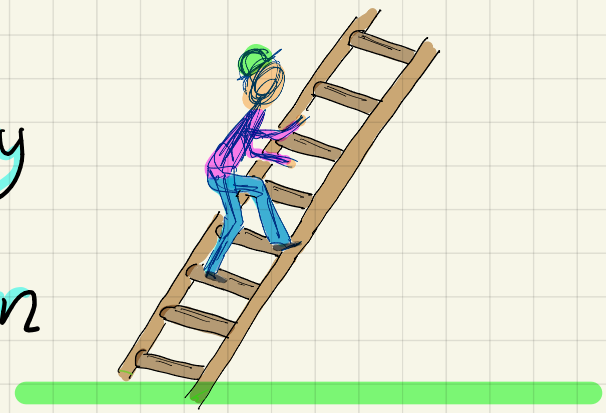
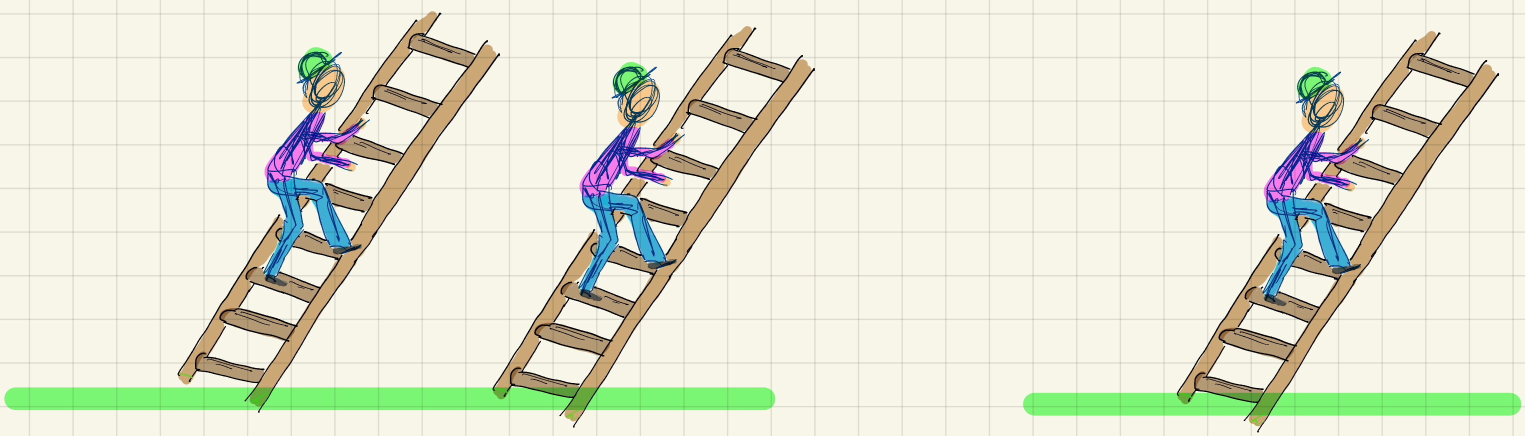




Proofs by Induction





Strong Induction

Base case Prove $P(k)$ true for $k \leq n_0$

Inductive Step: $\forall k \geq n_0 . \underbrace{\bigwedge_{i \leq k} P(i)}_{\text{Inductive hypo.}} \Rightarrow P(k+1)$

In other words, assume the property is true **up to** k, then prove it's also true for $k+1$.

- Note: It's typical that we won't need all statements up to P_k to be true, but only some of them.

The notation $\bigwedge_{i \leq k} P(i)$ means $P(k) \wedge P(k-1) \wedge P(k-2) \wedge \ldots$

Example 1 : For every $n \geq 12$, $n = 3x + 7y$ where $x, y \in \mathbb{N}$

Base case: $P(12) : 12 = 3(4) + 7(0)$

$\vdots$

Inductive step: $P(12), P(13), P(14), \ldots, P(k)$ are true

$P(k+1) : k+1 = 3x + 7y$

$$k+1 = (k+1) - 3 + 3 = \underbrace{k-2} + 3$$

$$= 3x' + 7y' + 3$$

$$= 3(x'+1) + 7y'$$

$$= 3x + 7y$$

Proof works when $k-2 \geq 12 \Rightarrow k \geq 14$. So $n_0 = 14$.

Example 2. Every $n \geq 1$ can be written as $n = m.2^i$ where m is odd.

Base case: $1 = 1.2^0$ ✓

Inductive Step: $P(1), P(2), P(3), \ldots, P(k)$ are true

$P(k+1): \ k+1 = m.2^i$

$k+1$ odd : $k+1 = (k+1)\,2^0$

$k+1$ even $\Rightarrow$ $k+1 = 2j$ where $j < k$?

so $P(j)$ is true and $j = m.2^l$

Therefore, $k+1 = 2[m.2^l] = m.2^{l+1} = m.2^i$.

Proof works as long as $j < k \Rightarrow \frac{k+1}{2} < k \Rightarrow k+1 < 2k \Rightarrow k > 1$.
So $n_0 = 1$.

Example 3:

Consider
$$a_1 = 3$$
$$a_2 = 5$$
$$a_n = 3a_{n-1} - 2a_{n-2}, \quad n \geqslant 3$$

Let's find a few a_n's:

$$a_3 = 3a_2 - 2a_1 = 3\cdot 5 - 2\cdot 3 = 15 - 6 = 9$$
$$a_4 = 3a_3 - 2a_2 = 3\cdot 9 - 2\cdot 5 = 27 - 10 = 17$$
$$a_5 = 3a_4 - 2a_3 = 3\cdot 17 - 2\cdot 9 = 51 - 18 = 33$$
$$\vdots$$

Guess $a_n =$

Prove $a_n = 2^n + 1$ for all $n \geqslant 3$

<u>Base case</u> $P(1)$: $a_1 = 2^1 + 1 = 3$ ✓

$\vdots$

<u>Inductive step:</u> $P(1), P(2), P(3), \ldots, P(k)$ are true

$P(k+1)$: $a_{k+1} = 2^{k+1} + 1$

$$a_{k+1} = 3a_k - 2a_{k-1} = 3\cdot(2^k+1) - 2(2^{k-1}+1)$$
$$= 3\cdot 2^k - 2\cdot 2^{k-1} + 1$$
$$= 3\cdot 2^k - 2^k + 1$$
$$= 2\cdot 2^k + 1$$
$$= 2^{k+1} + 1$$

Proof works if $k-1 \geqslant 1 \Rightarrow k \geqslant 2$. So $n_0 = 2$.

Example 4. Fibonacci Sequence

$$F_0 = 0$$

$$F_1 = 1$$

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2$$

Prove $F_n = \dfrac{1}{\sqrt{5}}\left[\phi^n - (1-\phi)^n\right]$ for $n \geq 0$

where $\phi = \dfrac{1+\sqrt{5}}{2}$ (ϕ is called the golden ration)

Note: Both ϕ and $1-\phi$ are solutions to $\dfrac{1}{x} + \dfrac{1}{x^2} = 1$

Base case: $P(0)$: $0 = \dfrac{1}{\sqrt{5}}\left[\phi^0 - (1-\phi)^0\right] = \dfrac{1}{\sqrt{5}}[1-1] = 0$ ✓

$P(1)$: $1 = \dfrac{1}{\sqrt{5}}\left[\phi - (1-\phi)\right] = \dfrac{1}{\sqrt{5}}(2\phi - 1) = 1$ ✓

Inductive step: $P(0), P(1), \ldots, P(k)$ are true.

$$P(k+1): \quad F_{k+1} = \frac{1}{\sqrt{5}}\left[\phi^{k+1} - (1-\phi)^{k+1}\right]$$

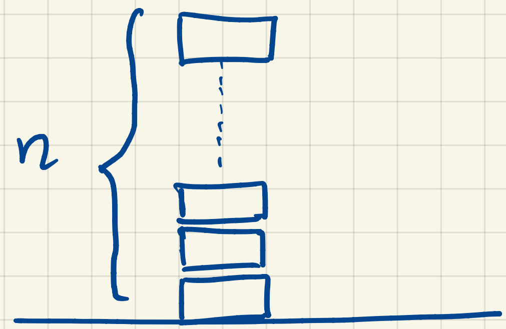
$$F_{k+1} = F_k + F_{k-1} = \frac{1}{\sqrt{5}}\left[\phi^{k} - (1-\phi)^{k}\right] + \frac{1}{\sqrt{5}}\left[\phi^{k-1} + (1-\phi)^{k-1}\right]$$

$$= \frac{1}{\sqrt{5}}\phi^{k+1}\left[\frac{1}{\phi} + \frac{1}{\phi^2}\right] - \frac{1}{\sqrt{5}}(1-\phi)^{k+1}\left[\frac{1}{1-\phi} + \frac{1}{(1-\phi)^2}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\phi^{k+1} - (1-\phi)^{k+1}\right]$$

Proof works if $k-1 \geqslant 0 \Rightarrow k \geqslant 1$. So $n_0 = 1$.

Consider a game with n blocks stacked in a tower



The goal is to split the stack repeatedly until we have n stacks of height 1.

Prove that for all $n \geq 1$, we need exactly $n-1$ splits.

Base case: $P(1)$: A stack of 1 block needs $1-1=0$ splits ✓

Inductive step: $P(1), P(2), P(3), \ldots, P(k)$ are true.

$P(k+1)$: A stack of $k+1$ blocks requires k splits.

<u>Make a move:</u> First split make two stacks of size s and $k+1-s$

observations: $k \geqslant 1 \Rightarrow k+1 \geqslant 2 \Rightarrow$ first split exists.

$$1 \leqslant s < k+1$$

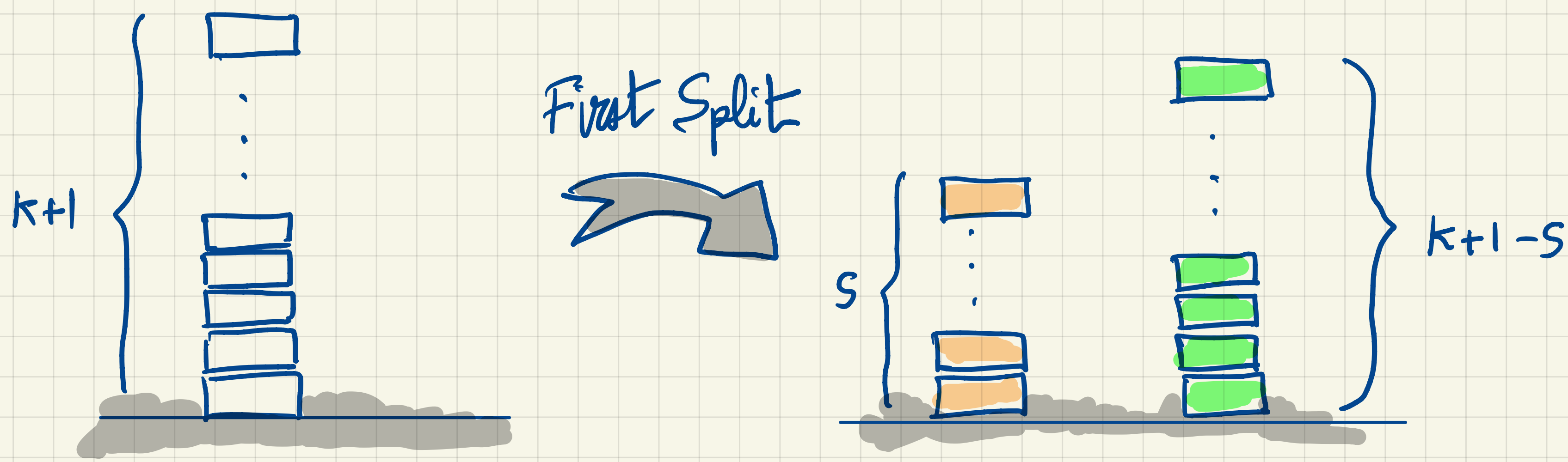
$$k+1-s < k+1$$

So $P(s)$ and $P(k+1-s)$ are true.

$$\text{Total number of splits} = 1 + (s-1) + (k+1-s-1)$$

$$= 1 + s - 1 + k + 1 - s - 1$$

$$= k.$$

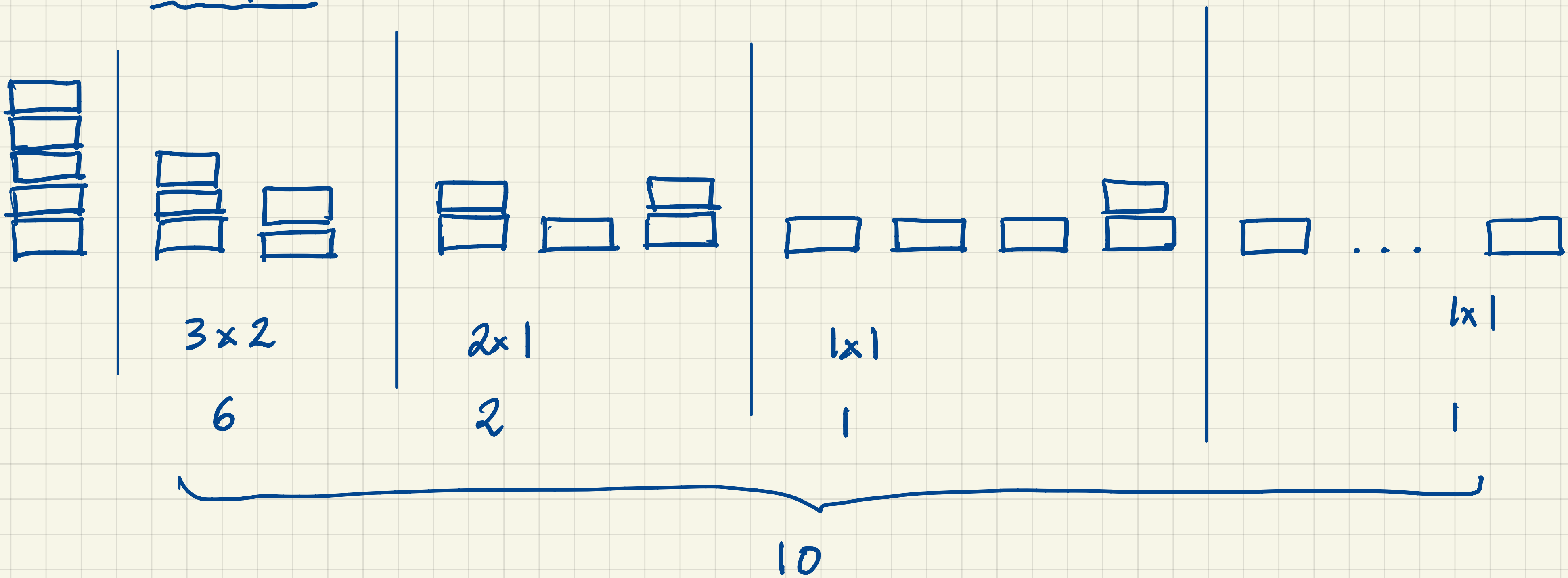


$$1 \leq s < k+1$$

$$1 \leq k+1-s < k+1$$

Variation: Assume that if you split n into s and $n-s$ you receive a score of $n(n-s)$.

Example:



Prove that the score is always $\binom{n}{2}$

<u>Base case:</u> $P(1)$: $\binom{1}{2} = 0$ ✓

<u>Inductive step:</u> $P(1), P(2), \ldots, P(k)$ are true.

$P(k+1)$: score is $\binom{k+1}{2}$

For $k+1$, the score is

$$\underbrace{s(k+1-s)}_{\text{First split}} + \binom{s}{2} + \binom{k+1-s}{2}$$

$$= s(k+1-s) + \frac{s(s-1)}{2} + \frac{(k+1-s)(k-s)}{2}$$

$$\vdots$$

$$= \frac{k(k+1)}{2} = \binom{k+1}{2}$$