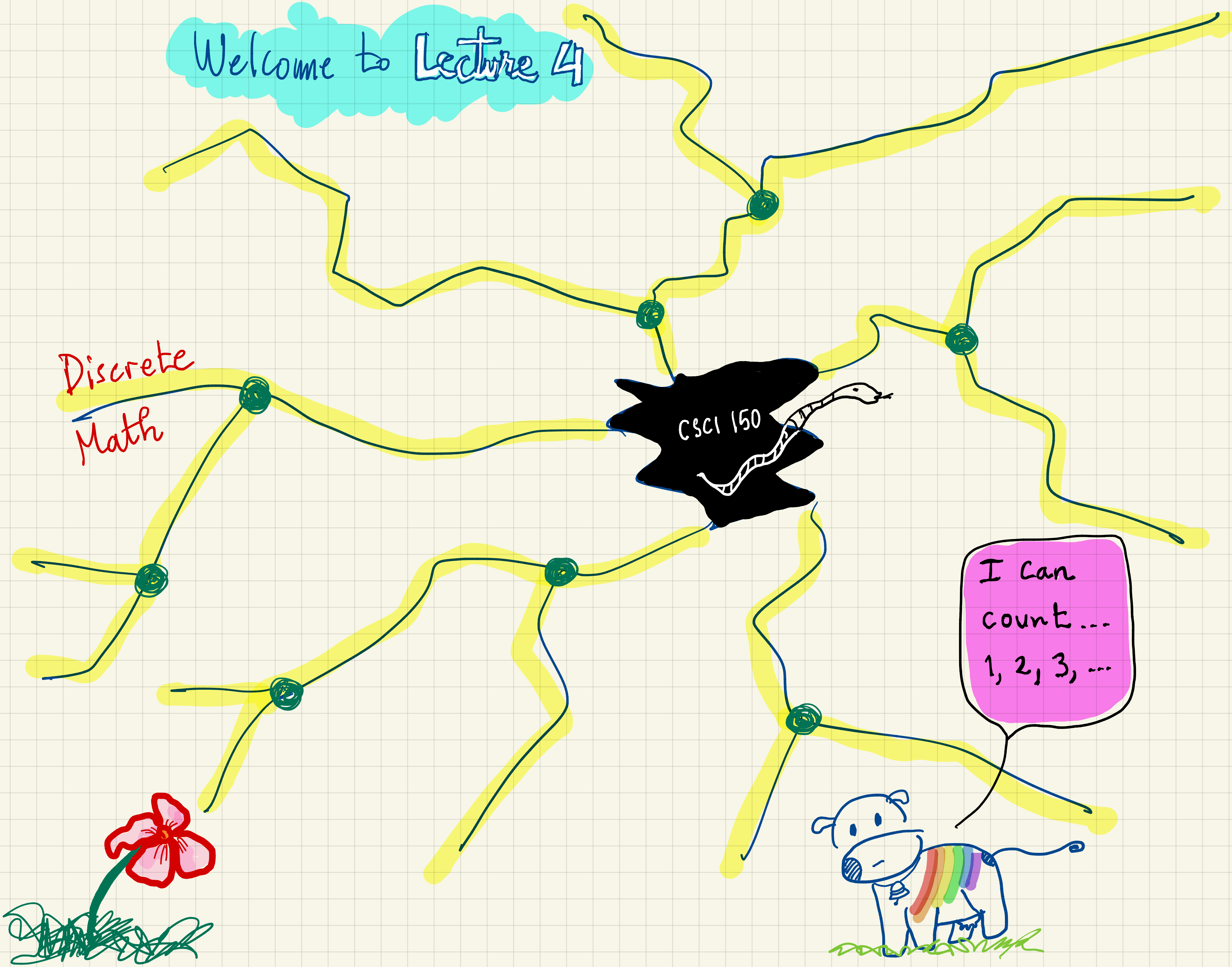
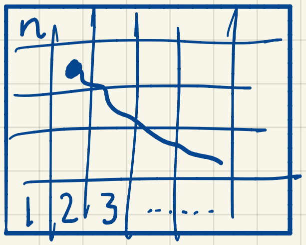

Welcome to Lecture 4
Discrete Math
CSCI 150
I can count...
1, 2, 3, ...

$\binom{n}{2}$ "themes"



- Two squares define a snake
- How many possible snakes?
- In how many ways can we select 2 squares out of n?

$$\binom{n}{2} = \frac{n(n-1)}{2} \quad \left(\text{also} = 1+2+\cdots\cdots+(n-1)\right)$$

Product rule:

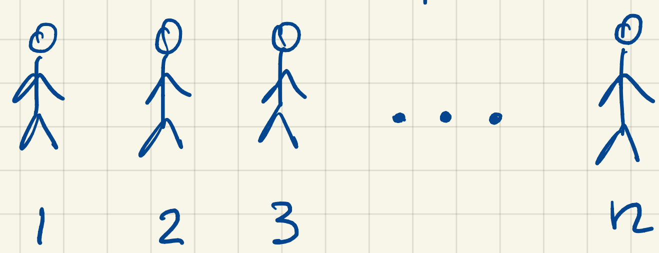
1. choose a square ----- n ways
2. choose another square --- (n-1) ways

$$n(n-1)$$

overcounting by 2. Answer $n(n-1)/2$.

Abstracting the idea of pairs ...

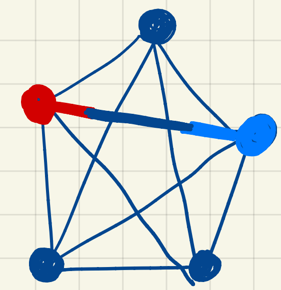
Let's say we have n people



And they all shook hands, how many handshakes do we count?

A handshake is defined by 2 people. so again, this is equivalent to the number of pairs $\binom{n}{2} = \frac{n(n-1)}{2}$

Given a graph with n vertices and all possible edges



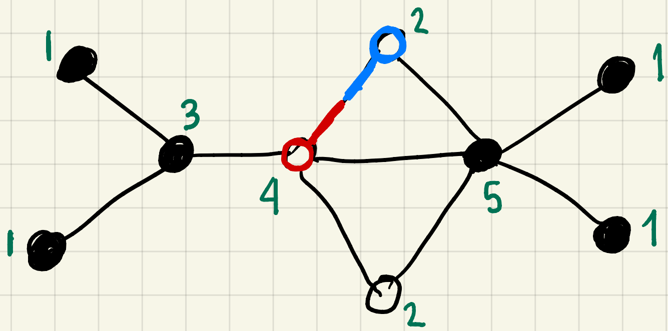
How many edges are there?

Again, $\binom{n}{2}$

$n=5$: $\binom{5}{2} = \frac{5 \times 4}{2} = 10$

Every vertex "sees" $(n-1)$ edges so $n(n-1)$, but we overcount by 2 because every edge is counted twice, once from each end.

Generalization: The handshake Lemma



<u>Degree of vertex</u>

edges touching it

Add up all degrees: $1+1+3+4+2+2+5+1+1 = 20$

edges $= 10$

Let d_i = degree of vertex i, given n vertices

$$d_1 + d_2 + \cdots + d_n = \sum_{i=1}^{n} d_i = 2 \cdot \#\text{ edges}$$

Other conclusions: $\sum_{i=1}^{n} d_i$ is always even

odd-degree vertices is always even

(counting establishes structure)

"n choose 2" $\binom{n}{2}$ in a set setting.

Given a set S with n elements

e.g. $S = \{1, 2, 3, \dots, n\}$

How many subsets of S contain two elements?
In other words, what's the number of subsets of size 2? Also $\binom{n}{2}$

Example: $S = \{1, 2, 3, 4\}$ and $\binom{4}{2} = \frac{4 \times 3}{2} = 6$

There are 16 subsets (including the empty subset)

try to find them

but the subsets of size 2 are

$\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$

So $\binom{n}{2}$ is the number of ways we can select 2 out of n <u>without</u> order.

Boys & Girls

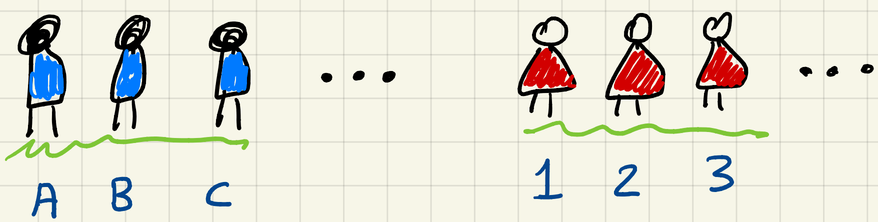
m boys

n girls

In how many ways we can make a couple? (a boy & a girl)

1. choose a boy .... m ways
2. Choose a girl .... n ways

$m \times n$



(No overcounting)

(B,3) for instance can only be generated in one way.

We can't permute choices of phase 2 and phase 1 (Different sets)

1st phase always generates a boy

2nd phase always generates a girl.

What if we want to choose any 2 people in the previous example?

1. Choose a person .... $m+n$ ways
2. Choose <u>another</u> person ..... $m+n-1$ ways

$$(m+n)(m+n-1)$$

Is there overcounting?

Yes, by 2. So $\dfrac{(m+n)(m+n-1)}{2} = \dbinom{m+n}{2}$

Using the addition rule, we conclude

$$\underbrace{\binom{m}{2}}_{\text{two boys}} + \underbrace{\binom{n}{2}}_{\text{two girls}} + \underbrace{m \cdot n}_{\text{a boy \& a girl}} = \underbrace{\binom{m+n}{2}}_{\text{any two}}$$

Disjoint categories

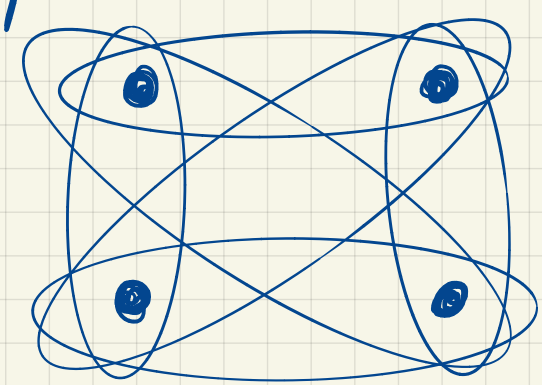
Be careful how we look at pairs:

$$\binom{n}{2} \neq \frac{n}{2}$$

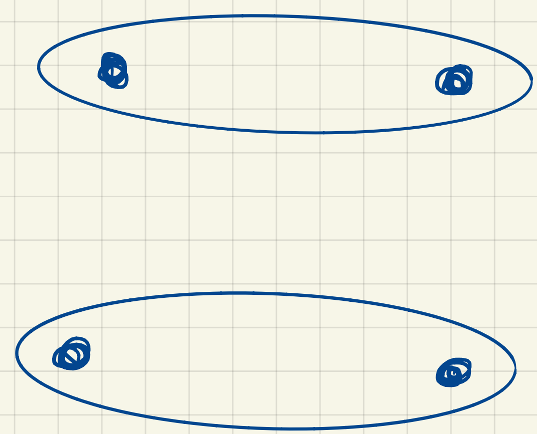
$\binom{n}{2}$: In how many ways we can select a pair or equivalently all possible pairs

$\frac{n}{2}$: (when n even) number of pairs that can exist simultaneously

Example: $n=4$

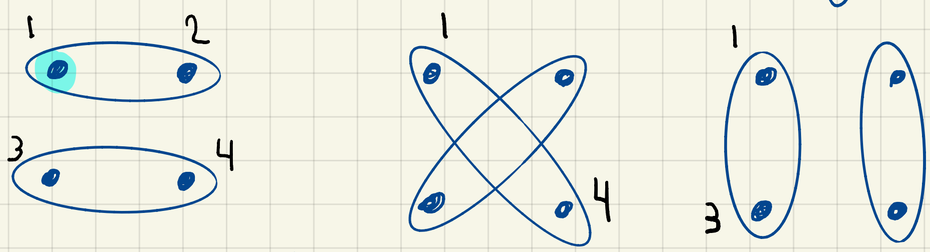


$\binom{4}{2}=6$



$\frac{4}{2}=2$

This leads to another question.
In how many ways can we make simultaneous pairs; e.g. teams of 2.



When 4 people, the answer is 3.

That's not $\binom{4}{2}$ and not $\frac{4}{2}$

What is it?

Assume we have $2n$ people (to make it even). In how many ways can we make n teams?

Team 1
1. choose a person .... $2n$ ways
2. choose another .... $2n-1$ way

Team 2
3. choose another .... $2n-2$ ways
4. choose another .... $2n-3$ ways

⋮

Team n
$2n-1$. choose another .... 2 ways
$2n$. choose another .... 1 way

$(2n)!$ $\neq 2n!$

$n=2 \Rightarrow 4! \neq 3$

Let's look at overcounting.

- Permuting choices within a team results in the same outcome, that's an overcount of 2 per team
- Permuting teams can be done in $n!$ ways.

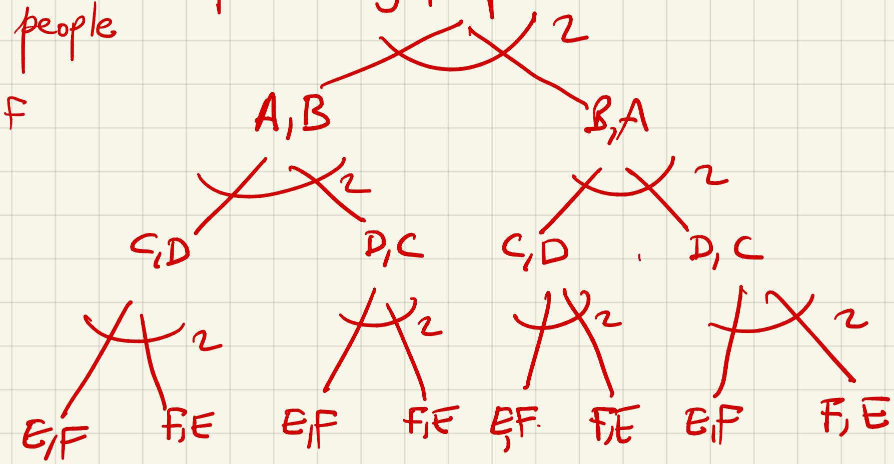
Total overcount is $2^n n!$

Answer: $\dfrac{(2n)!}{2^n n!}$ try $n=2$. $\dfrac{4!}{2^2 \times 2!} = \dfrac{4 \times 3}{4} = 3$ ✓

Example: $n=3 \Rightarrow 6$ people

A, B C, D E, F

permuting people with team



$2 \times 2 \times 2 = 2^3$, In general 2^n.

permuting teams

A, B	C, D	E, F
A, B	E, F	C, D
C, D	A, B	E, F
C, D	E, F	A, B
E, F	A, B	C, D
E, F	C, D	A, B

6 permutations $= 3!$, In general $n!$

Another way.

1. Choose 2 people ..... $\binom{2n}{2}$ ways
2. Choose another 2 people .... $\binom{2n-2}{2}$ ways

$\vdots$

n. Choose another 2 people .... $\binom{2}{2}$ ways

$$\binom{2n}{2}\binom{2n-2}{2}\binom{2n-4}{2} \cdots\cdots \binom{2}{2}$$

Now we don't overcount due to permutations within team, because the "choose 2" takes care of that.
We can still permute the n teams in n! ways, so we are overcounting by that much.

Answer: $$\frac{\binom{2n}{2}\binom{2n-2}{2}\binom{2n-4}{2}\cdots\cdots\binom{2}{2}}{n!} = \frac{\prod_{i=0}^{n-1}\binom{2n-2i}{2}}{n!}$$

Exercices: • Verify $\prod_{i=0}^{n-1} \binom{2n-2i}{2}$ has exactly n terms in the prod, and generates

$$\binom{2n}{2}\binom{2n-2}{2}\binom{2n-4}{2}\cdots\binom{2}{2}$$

• Using $\binom{n}{2} = \frac{n(n-1)}{2}$, verify that above is equal to $\frac{(2n)!}{2^n}$

• Using $\binom{n}{2} = \frac{n(n-1)}{2}$, verify that

$$\binom{m}{2} + \binom{n}{2} + mn = \binom{m+n}{2}$$