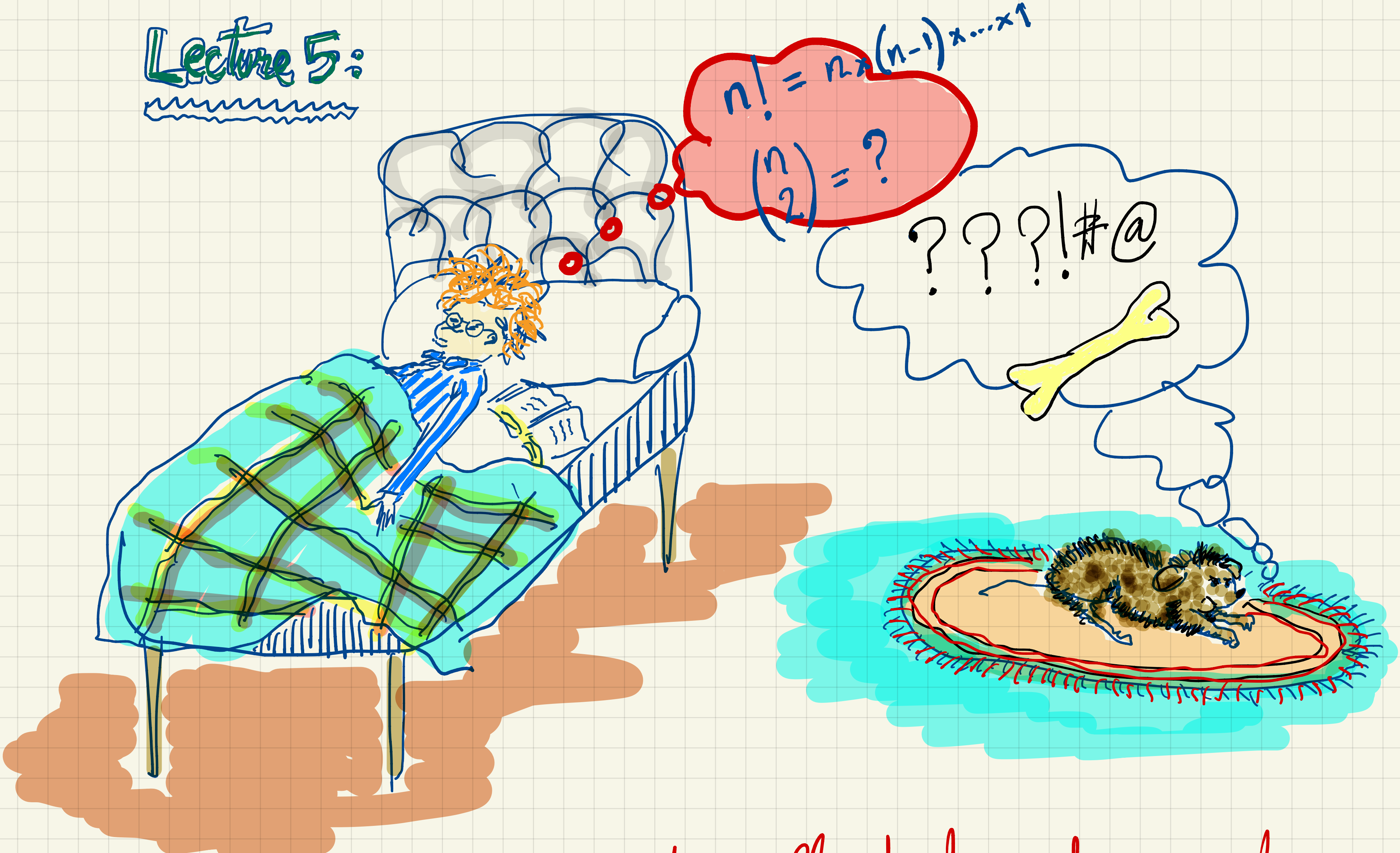
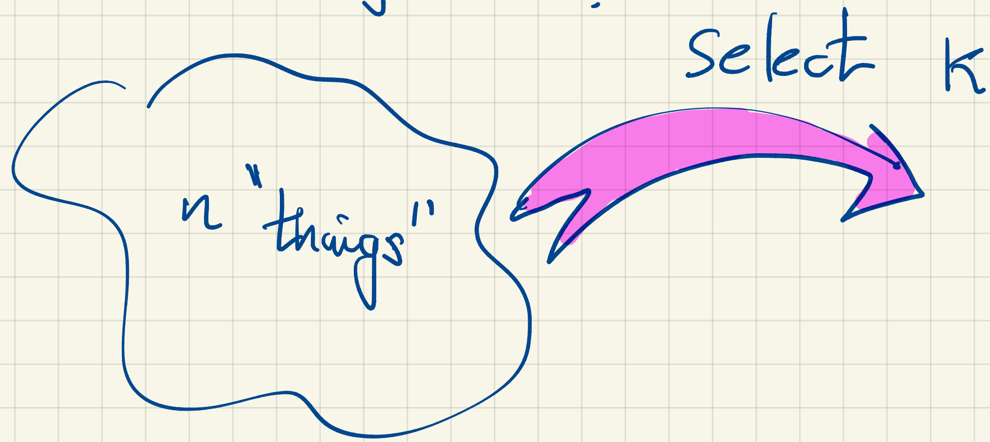
Lecture 5:



We will start in few minutes...

Question: How many ways we can select k out of n?



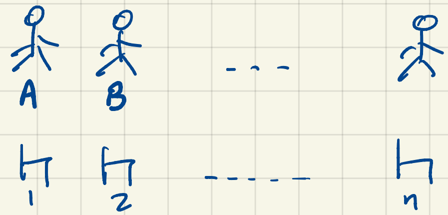
- Order is relevant

- Order is not relevant

<u>Example:</u> $\binom{n}{2}$ = # ways we can select 2 out of n with <u>no order</u>

n people

n chair



In how many ways can I seat people on chairs?

High level thinking: It's just a permutation $\Rightarrow n!$

"Primitive way":

1. choose a person for chair 1 ...... n ways
2. choose a person for chair 2 ...... $(n-1)$ ways

⋮

n. choose a person for chair n ...... 1 way

product rule: $n(n-1)(n-2)\cdots 1 = n!$

(This procedure does not overcount)

Another procedure:

1. choose a person .... n
2. choose a chair for that person ..... n
3. choose another person .... $(n-1)$
4. choose another chair ...... $(n-1)$

$\vdots$

$2n-1$. choose another person 1

$2n$. choose another chair 1

$$n! \times n! = (n!)^2$$

This procedure **overcounts** by **$n!$**

Ex: $n = 3$

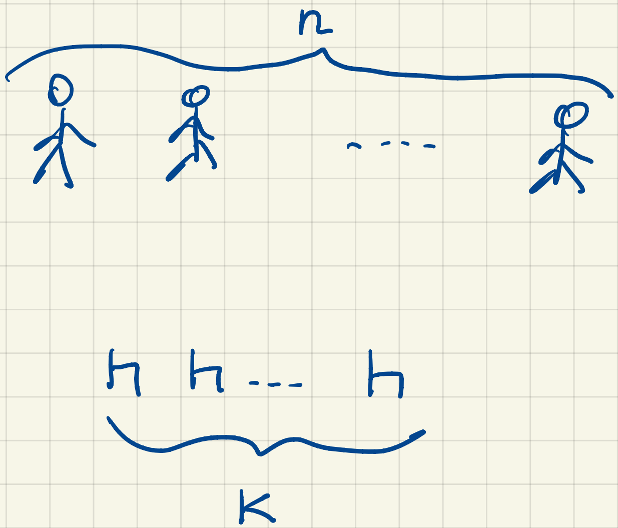
(A, 1) (B, 2) (C, 3)

(B, 2) (C, 3) (A, 1)

In general, we have $n!$ permutation that are equivalent.

n people

k chairs



$k \le n$

The procedure can be generalized to this setting where $k \le n$.

1. choose a person for chair 1 ...... n ways $\quad n-(1-1)$
2. choose another person for chair 2 .... $(n-1)$ way $\quad n-(2-1)$

$\vdots$

k. choose another person for chair k .... $\underbrace{\boxed{n-(k-1)}}_{n-k+1}$ ways

Product rule: $n(n-1)(n-2)\cdots\cdots(n-k+1)$

example: $k=n \Longrightarrow n(n-1)(n-2)\cdots(1) = n!$

So far: (If I were to continue)

$$\frac{n(n-1)\cdots\cdots(n-k+1).\overbrace{(n-k)(n-k-1)\cdots 1}^{(n-k)!}}{(n-k)!} = \boxed{\frac{n!}{(n-k)!}} = \prod_{i=0}^{k-1}(n-i)$$

Conclusion: In how many ways I can select k out of n with order

The number of k-permutations

Example:

$$k=n \Rightarrow \frac{n!}{(n-k)!} = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$$

$$k=1 \Rightarrow \frac{n!}{(n-k)!} = \frac{n!}{(n-1)!} = \frac{n\times\cancel{(n-1)\times\cdots\times 1}}{\cancel{(n-1)\times\cdots\cdots\times 1}} = n$$

$$k=0 \Rightarrow \frac{n!}{(n-k)!} = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$$

What if it does not matter where someone sits as long as they sit?

<u>Example:</u>

$k = 3$

$n \geq 3$

$(A,1)\ (B,2)\ (C,3)$

$(A,1)\ (C,2)\ (B,3)$

$(B,1)\ (A,2)\ (C,3)$

$(B,1)\ (C,2)\ (A,3)$

$(C,1)\ (A,2)\ (B,3)$

$(C,1)\ (B,2)\ (A,3)$

Now Equivalent.

My previous procedure overcounts by # ways I can permute the choices of k phases $\Rightarrow k!$

$$\binom{n}{k} = \boxed{\frac{n!}{k!\,(n-k)!}}$$

ways I can select k out of n without order

"n choose k"

$$\frac{n!}{(n-k)!} \geq \binom{n}{k}$$

ordered $\qquad\qquad$ unordered

$$\binom{n}{k} = \boxed{\frac{n!}{k!\,(n-k)!}}$$

$k = 2$: ? $\qquad \dfrac{n!}{2!\,(n-2)!} = \dfrac{n(n-1)\cancel{(n-2)}\cdots\cancel{1}}{2\cancel{(n-2)}\cdots\cancel{1}} = \dfrac{n(n-1)}{2}$ ✓

$k = 1$: ? $\qquad \dfrac{n!}{1!\,(n-1)!} = n.$

$k = 0$: ? $\qquad \dfrac{n!}{0!\,(n-0)!} = \dfrac{n!}{1 \cdot n!} = 1.$

What if I want to select k out of n with order, but I can repeat my choice? ($k \leq n$ or $k \geq n$)

1. choose one ------- n ways
2. choose one ------- n ways

$\vdots$

k. choose one ------- n ways

n^k ways

Example: How many 7 letter words can I make using the alphabet $\{a, b, c\}$

$n = 3$, $k = 7$ Therefore 3^7.

Select k out of n

	no order	order
no repetition:	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$	$k!\binom{n}{k}$
repetition:	?	n^k

Later!