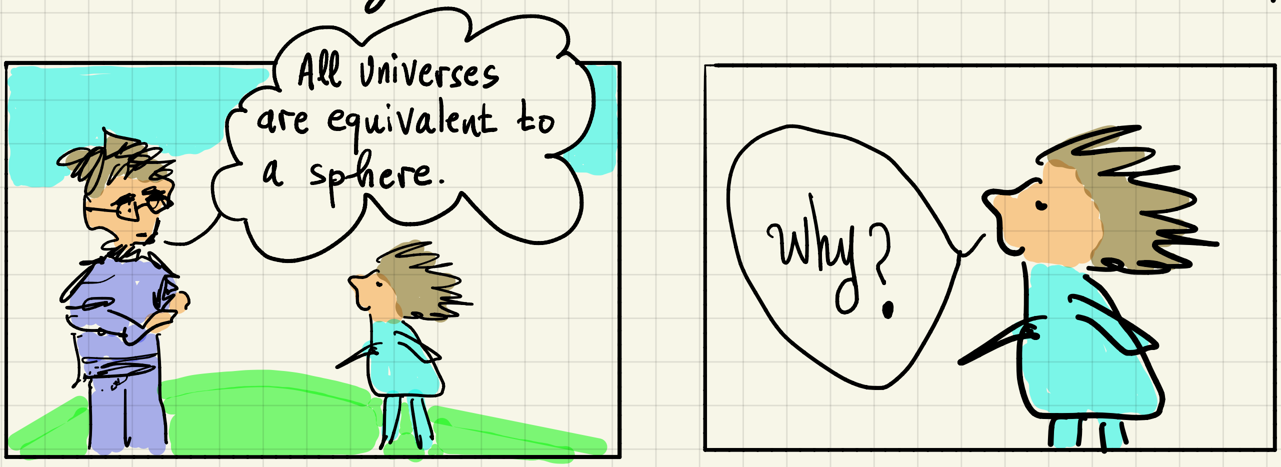
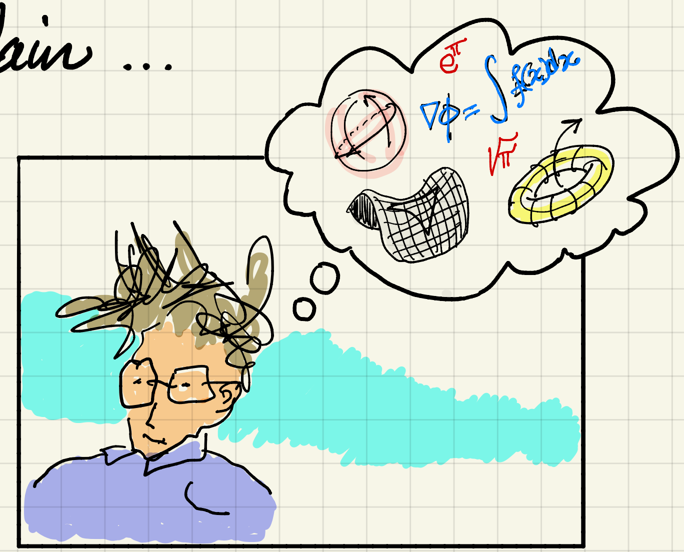
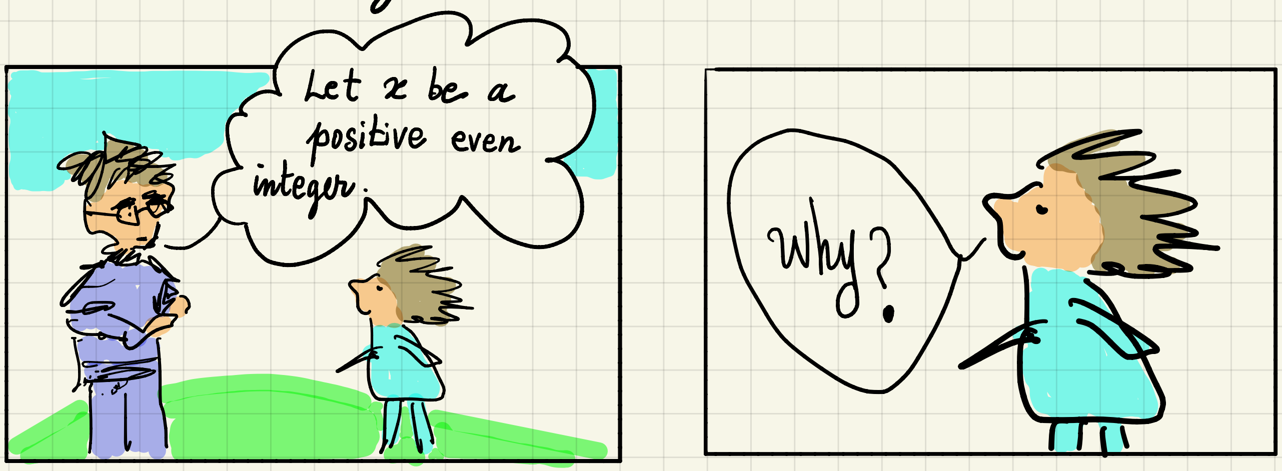
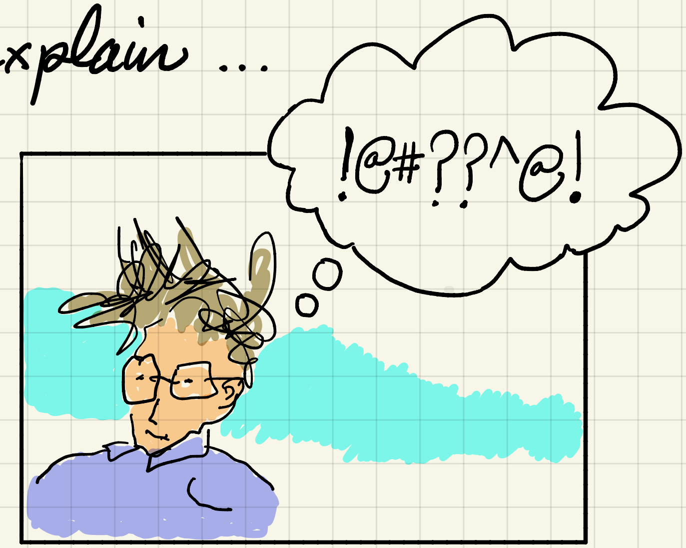
Things that mathematicians can explain ...





Things that mathematicians cannot explain ...



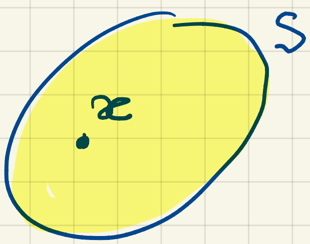


Welcome to **Lecture 6**

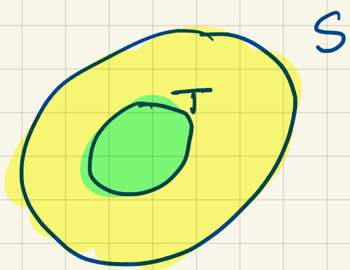
Sets, Relations, function
(and more ?...)

* A set is an unordered collection of elements

  ex: $S = \{x, y, z\} = \{y, x, z\}$

* 

  $x \in S$ x is an element of S

* 

  $T \subset S$ T is a subset of S (both are sets)

  Every element of T is an element of S

* $\phi = \{\ \}$ is the empty set

* $\phi \subset S$ $\qquad$ ϕ is a subset of every set (why?)

* $S \subset S$ $\qquad$ a set is a subset of itself

* $T = S$ means $T \subset S$ and $S \subset T$ (Equality)

* When $T \subset S$ and $T \neq S$, we say T is a <u>proper subset</u> of S

Negations (scratch the symbol)

$T \neq S$ "T is NOT equal to S"

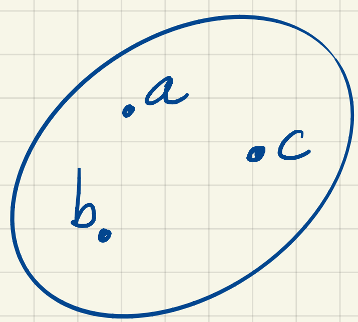
$x \notin S$ "x is NOT an element of S"

$T \not\subseteq S$ "T is NOT a subset of S"

Size of a set (also called cardinality)

When S is finite, $|S|$ is size of S

Ex: $S = \{a, b, c\}$



$|S| = 3$

Examples of sets of numbers

$$\mathbb{N} = \{1, 2, 3, \ldots\} = \{x \mid x \text{ is a positive integer}\}$$

"such that"

$$\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

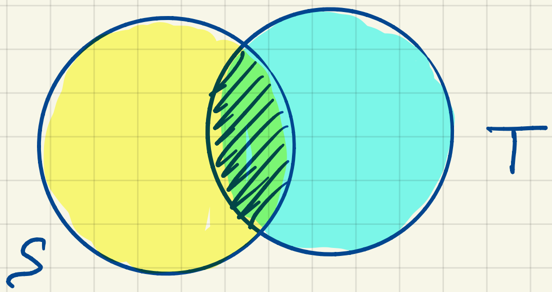
$$= \{x \mid x \text{ is an integer}\}$$

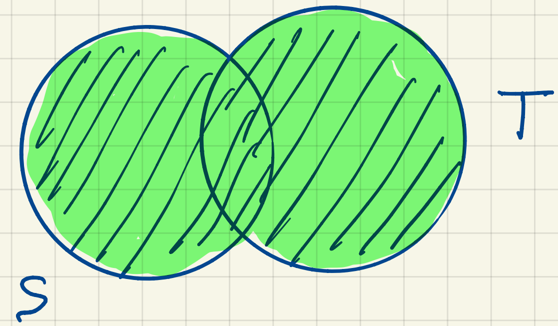
$$\mathbb{Q} = \{x \mid x = \frac{a}{b} \text{ where } a \in \mathbb{Z}, b \in \mathbb{N}\}$$

This is the set of rational numbers

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$$

Intersection & Union





$S \cap T = \{x \mid x \in S \text{ and } x \in T\}$

$S \cup T = \{x \mid x \in S \text{ or } x \in T\}$

not exclusive or

$$\cap$$

$$\cup$$

Product of two sets (also called Cartesian prod.)

* $S \times T = \{(x,y) \mid x \in S \text{ and } y \in T\}$ (Not symmetric)

  $S \times T \neq T \times S$

Ex: $S = \{a,b,c\}$ $T = \{1,2\}$

$$S \times T = \{(a,1),(a,2),(b,1),(b,2),(c,1),(c,2)\}$$

☞ : $(a,1) \neq (1,a)$ (but $\{a,1\} = \{1,a\}$)

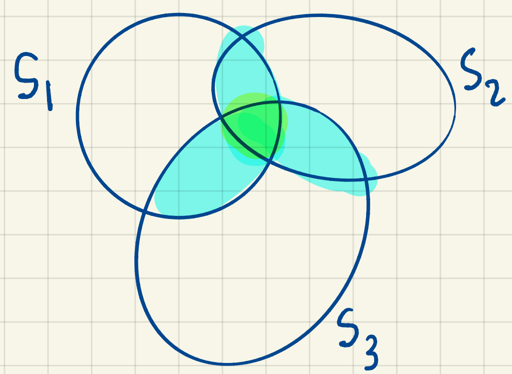
* $|S \times T| = |S| \times |T|$ (Recall product rule)

To generate a pair: 1. choose an element of S ..... $|S|$ ways

2. choose an element of T .... $|T|$ ways

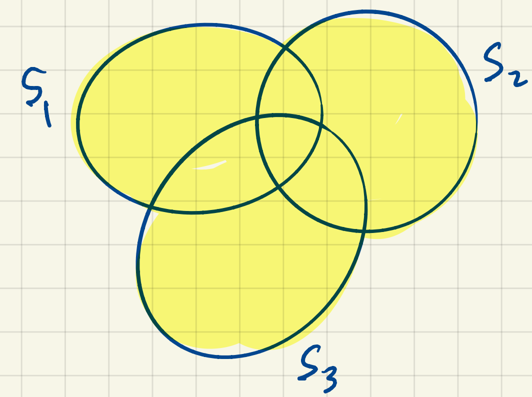
$|S| \times |T|$ ways

Intersection, Union, Product can be generalized to multiple sets



$S_1 \cap S_2 \cap S_3$

$\{x \mid x \in S_1 \text{ and } x \in S_2 \text{ and } x \in S_3\}$



$S_1 \cup S_2 \cup S_3$

$\{x \mid x \in S_1 \text{ or } x \in S_2 \text{ or } x \in S_3\}$

$$S_1 \times S_2 \times S_3 = \{(x, y, z) \mid x \in S_1 \text{ and } y \in S_2 \text{ and } z \in S_3\}$$

Side Remark: $S_1 \cap S_2 \cap S_3 \cap \ldots \cap S_n = \bigcap_{i=1}^{n} S_i$

(same for union)

Example product of 3 sets

$S = \{a, b, c\}$

$T = \{1, 2\}$

$R = \{\heartsuit, \diamondsuit\}$

$$S \times T \times R = \{ (a,1,\heartsuit), (a,1,\diamondsuit), (a,2,\heartsuit), (a,2,\diamondsuit),$$
$$(b,1,\heartsuit), (b,1,\diamondsuit), (b,2,\heartsuit), (b,2,\diamondsuit),$$
$$(c,1,\heartsuit), (c,1,\diamondsuit), (c,2,\heartsuit), (c,2,\diamondsuit) \}$$

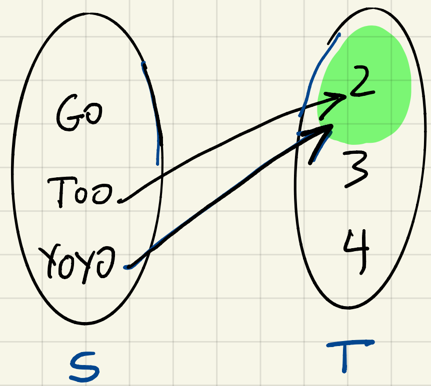
Relations

A relation from S to T is a subset of $S \times T$

Example: $S = \{GO, TOO, YOYO\}$

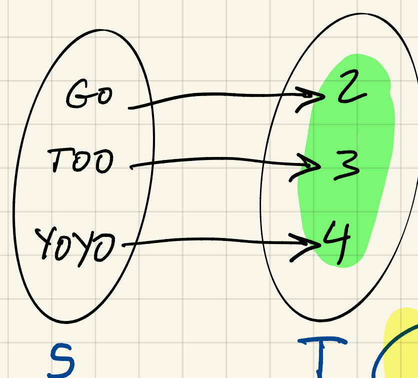
$T = \{2, 3, 4\}$

$R1 = \{(TOO, 2), (YOYO, 2)\}$



Not a function

$R2 = \{(GO, 2), (TOO, 3), (YOYO, 4)\}$

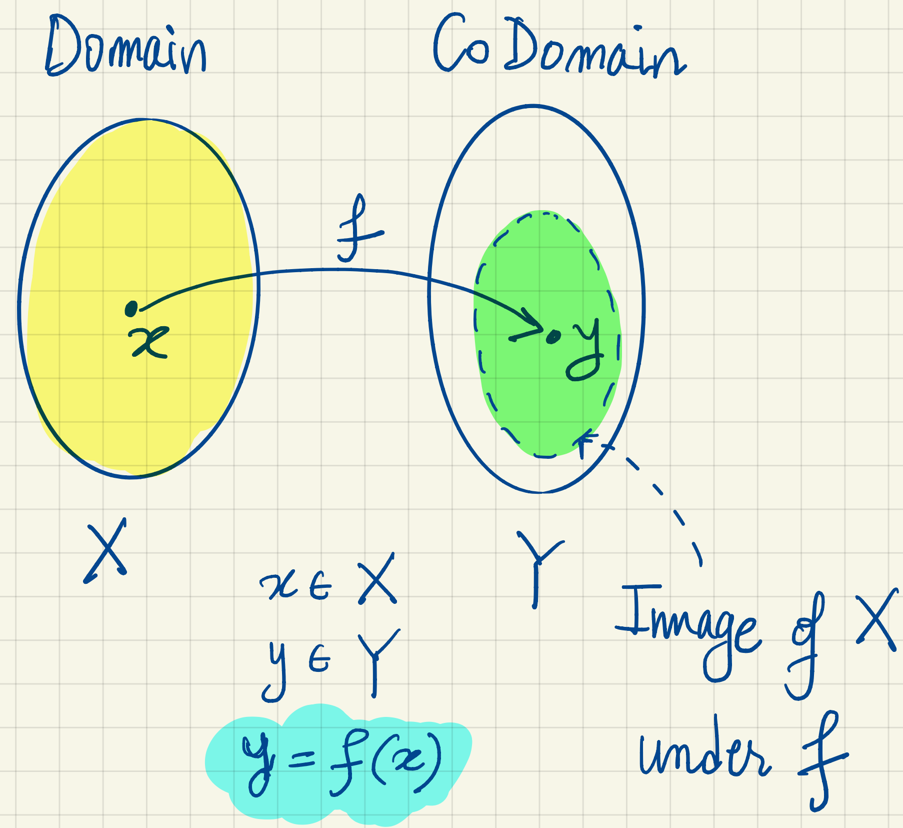


a function

Every element in S is mapped to exactly one element in T

Some notation (and definitions)

$$f : X \to Y$$



- A function that maps the elements of X to elements of Y

  X : Domain of function f.
  Y : CoDomain of function f

- Evaluating the function at each element of X produces the image of X (subset of Y in green)

- When the image is the entire set Y we call the function **onto**

Function : for **All** $x \in X$, there **Exists** only one $y \in Y$ such that $f(x) = y$

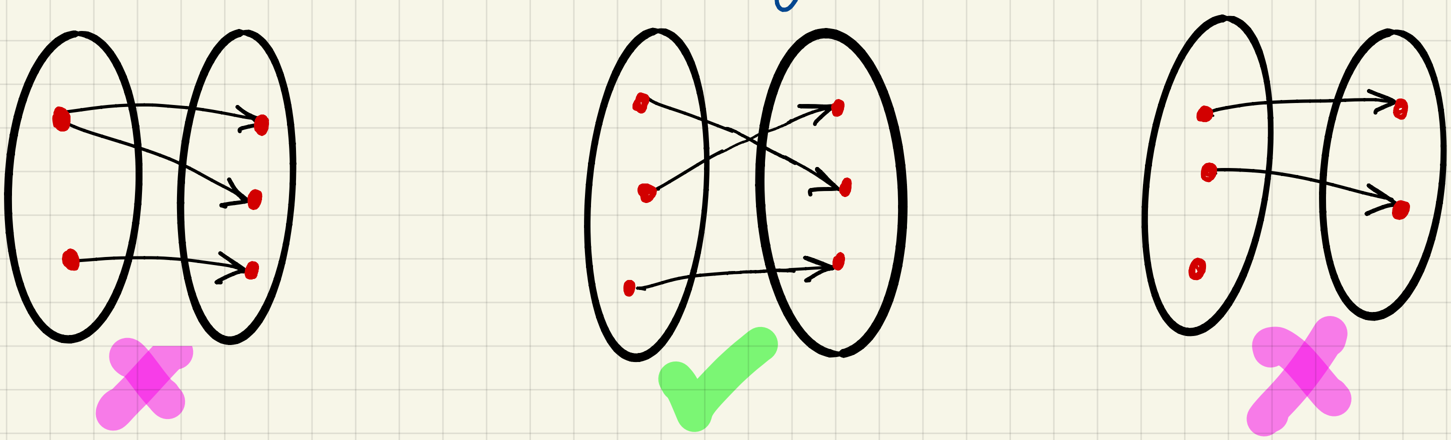
Notation: $\forall x \in X .\ \exists$ only one $y \in Y .\ f(x) = y$

for every element x of X

All $\longrightarrow$ $\forall$ (universal quantifier)

Exists $\longrightarrow$ $\exists$ (existential quantifier)

Exercice: which of the following are functions?



Exercice: Write the meaning of **onto** using $\forall$ and $\exists$.

Another property of functions:

One to one: For all x_1 and x_2 in X,

if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$

Notation: if A, then B $\longrightarrow$ $A \Rightarrow B$

(A implies B)

Exercice: Write definition of one to one using symbols in the following set.

$$\{ \forall, \exists, \Rightarrow \}$$

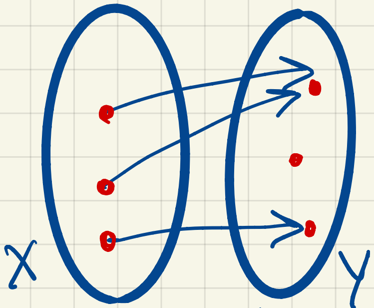
(You don't have to use all of them)

A functions that is both onto and one to one is called a **bijection**, or a **one to one correspondence**

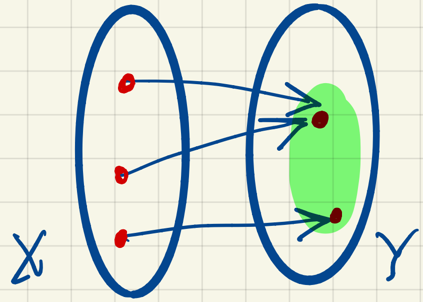
Exercice

	onto	one to one	bijection
$f: \mathbb{Z} \to \mathbb{N}$, $f(x) = \|x\|+1$	Yes	No	No
$g: \mathbb{N} \to \mathbb{Z}$, $g(x) = 2x$	No	Yes	No
$h: \mathbb{N} \to \{2,4,6,8,\dots\}$, $h(x) = 2x$	Yes	Yes	Yes

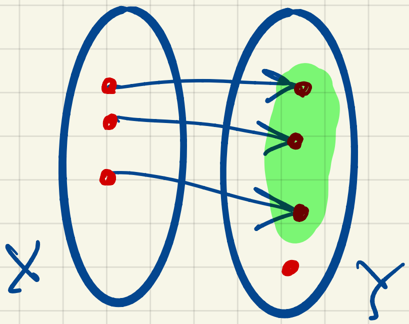
Illustrations



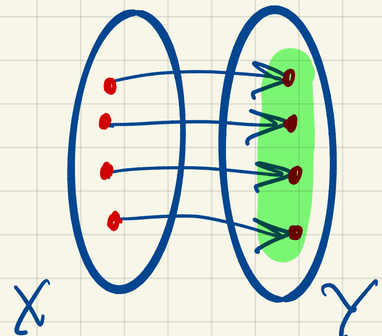
not onto
not one to one



Onto
not one to one



One to one
not onto



bijection

bijection means

$$|X| = |Y|$$

and we take this to mean "equal size" for infinite sets

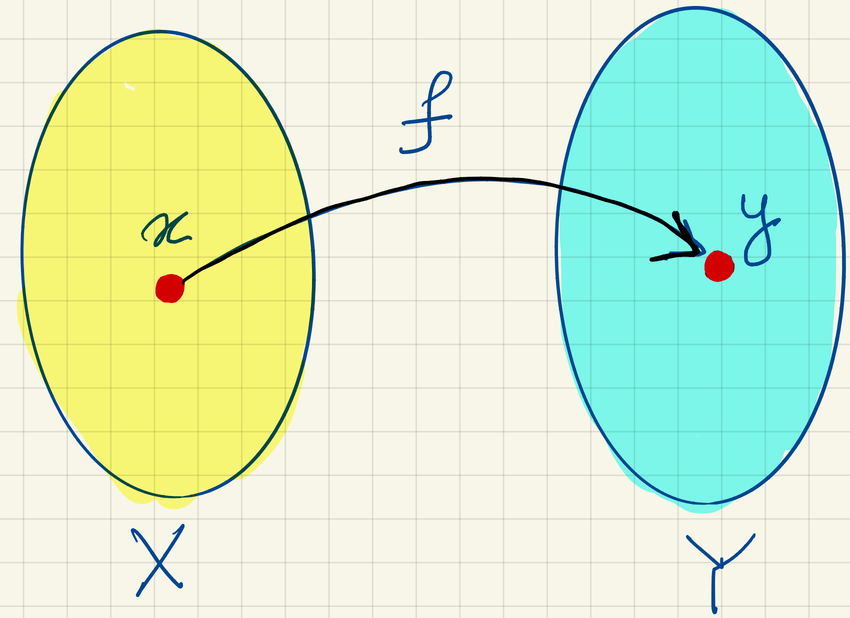
$$f : X \to Y$$

Onto : $\forall y \in Y .\ \exists x \in X .\ f(x) = y$

one to one : $\forall x_1 \in X .\ \forall x_2 \in X .\ (x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2))$

or $\forall x_1, x_2 \in X .\ (x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2))$

In simple terms



function: there is exactly one outgoing arrow for each x

1) onto: there is at least one incoming arrow for each y

2) one to one: there is at most one incoming arrow for each y

3) bijection: 1) and 2), there is exactly one incoming arrow for each y

Bijective function $\Longrightarrow |X| = |Y|$