
Lecture 7
We have n points on a circle...
How many points do we have?

Two simple problems on binary patterns

$$\underbrace{\boxed{0} \boxed{1} \boxed{1} \boxed{0} \boxed{0} \boxed{1} \cdots \boxed{1} \boxed{0}}_{n \text{ bits}}$$

How many words with n bits can we have ?

$S = \{0, 1\}$

Choose n times with <u>order</u> & <u>repetition</u>

$\Rightarrow \quad 2^n$

<u>Procedure</u>

1. choose a bit .... 2 ways
2. choose a bit ..... 2 ways

$\vdots$

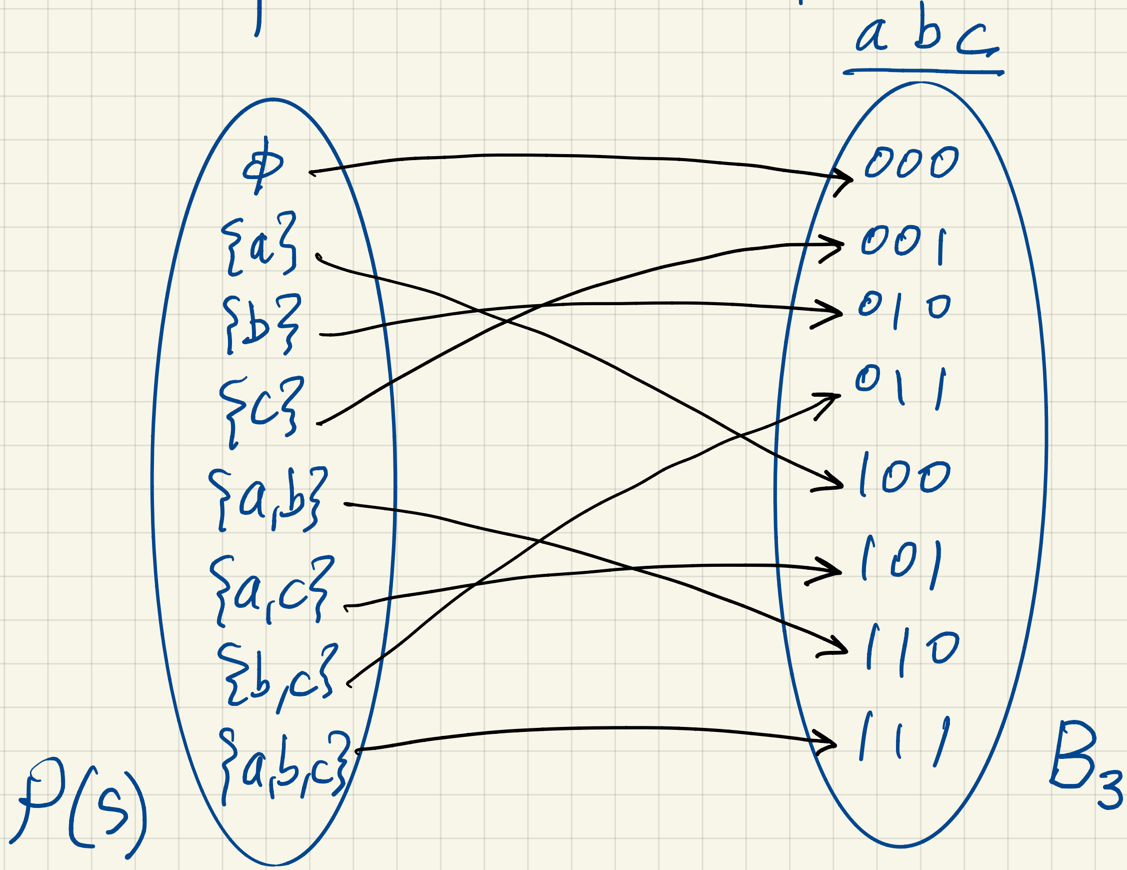
n. choose a bit ..... 2 ways

2^n ways

Similar to number of subsets of a set with n elements. Why?

Consider the set $S = \{a, b, c\}$

$$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$$

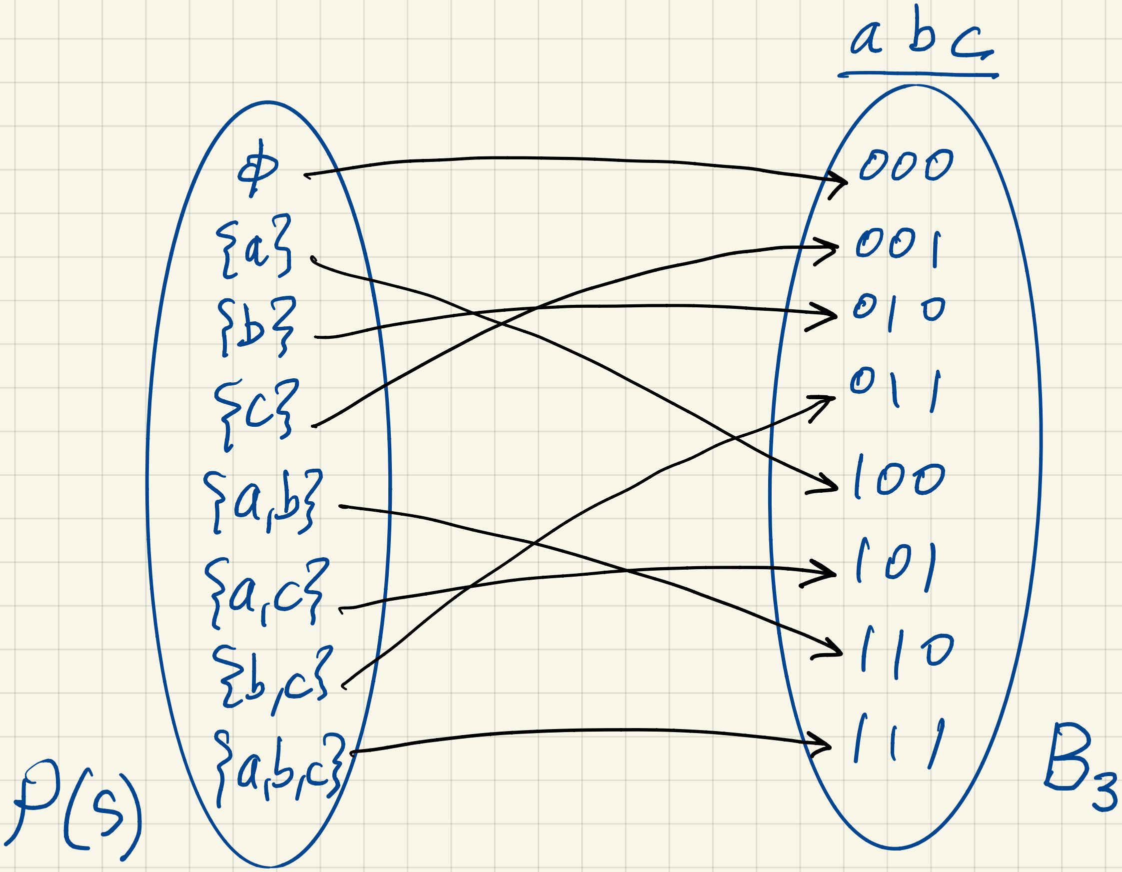


$f: P(S) \longrightarrow B_n$

$y = f(x)$

if i^{th} element $\in x$

set i^{th} bit of y to 1



$f: P(S) \longrightarrow B_n$

$y = f(x)$

if i^{th} element $\in x$

set i^{th} bit of y to 1

- $\forall\, x_1, x_2 \in P(x)$

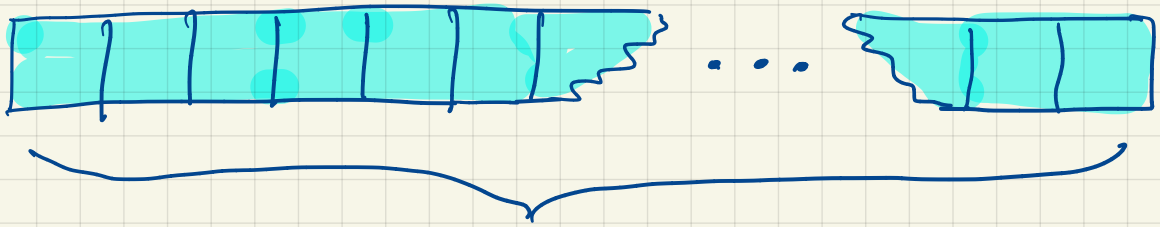
  $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$ [one-to-one]

- $\forall\, y \in B_n \,.\, \exists\, x \in P(S) \,.\, f(x) = y$ [onto]

Bijection

$|P(S)| = |B_n|$

Another Problem ....



n bits

$k \le n$

How many words with n bits have exactly k 1s ?

Choose k out of the n bits and make them 1s.

$$\Longrightarrow \binom{n}{k}$$

Remember this !!

<u>Example:</u> $n = 10$
$k = 3$

how many words of 10 bits have exactly 3 ones ?

$$\binom{10}{3} = \frac{10!}{3!\,(10-3)!} = \frac{10 \times 9 \times 8}{6}$$

Select k out of n

	no order	order
no repetition :	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$	$k!\binom{n}{k} = \frac{n!}{(n-k)!}$
repetition :	?	n^k

Today

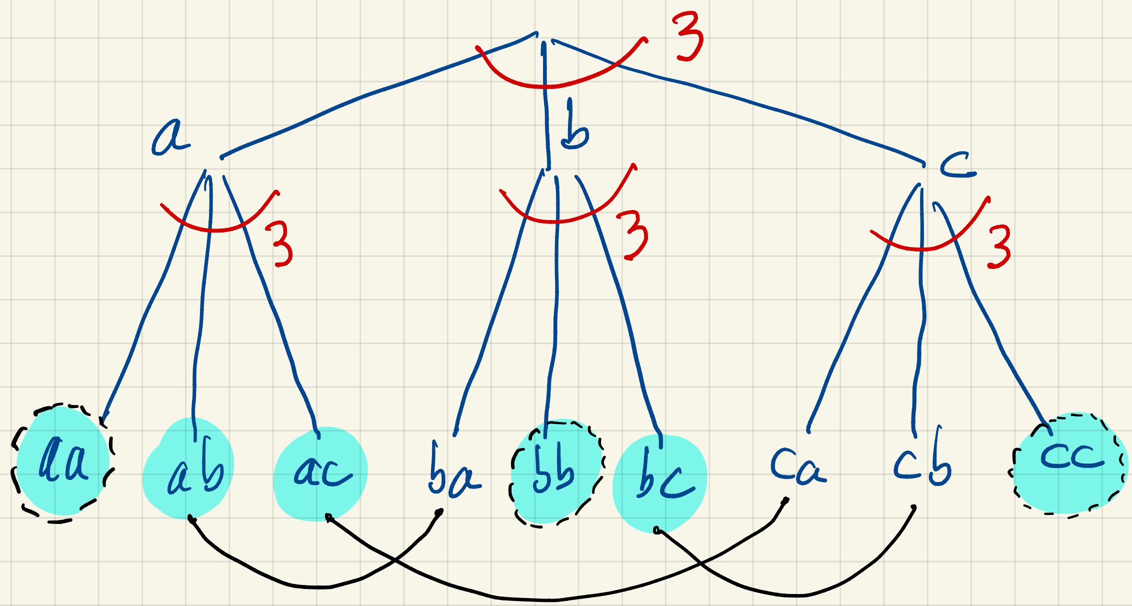
<u>Example:</u> lottery ticket where you can choose a number more than once !!

Example: $S = \{a, b, c\}$ $n = 3$

let $k = 2$ (choose 2 unordered with repetition)

aa	ab	ac
ba	bb	bc
ca	cb	cc

Failure of product rule



Some outcomes are over counted some are not!

Can't adjust from $3 \times 3 = 9$

Establish a bijection

Multisets of size 2

$\{b,a\} = \{a,b\}$

$$\begin{array}{ccc} & & a\ b\ c \\ \{a,a\} & \mapsto & (2,0,0) \\ \{a,b\} & \mapsto & (1,1,0) \\ \{a,c\} & \mapsto & (1,0,1) \\ \{b,b\} & \mapsto & (0,2,0) \\ \{b,c\} & \mapsto & (0,1,1) \\ \{c,c\} & \mapsto & (0,0,2) \end{array}$$

Set of solutions to:

$$x_1 + x_2 + x_3 = 2 \qquad x_i \geq 0$$

$x_i \in \{0,1,2,\dots\}$

Bijection

$n = 3$

$k = 2$

Generalize: How many solutions do we have

for: $x_1 + x_2 + \dots + x_n = k \qquad x_i \geq 0$?

$x_i \in \{0,1,2,3,\dots\}$

$S = \{a, b, c\}$ $\quad n = 3$

$k = 3$

$\{a,a,a\} \mapsto (3,0,0)$

$\{a,a,b\} \mapsto (2,1,0)$

$\{a,a,c\} \mapsto (2,0,1)$

$\{a,b,b\}$

$\{a,b,c\}$

$\{b,b,c\}$

$\{b,b,b\}$

$\vdots$

$\{c,c,c\}$ $\quad (0,0,3)$

Multisets of size 3

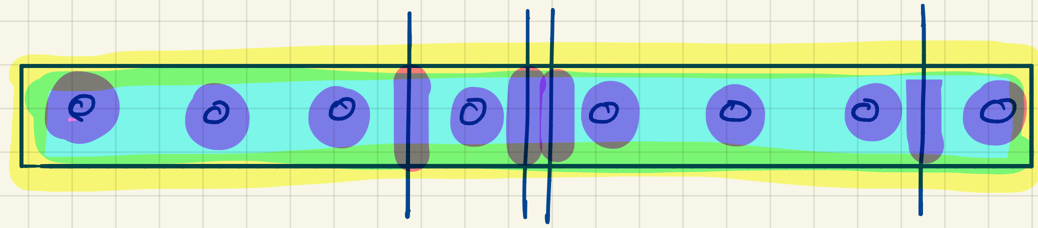
integer solutions to $x_1 + x_2 + x_3 = 3$, $x_i \geq 0$

We are essentially dividing k into n parts

$$x_1 + x_2 + \cdots + x_n = k$$

Ex: $k=8$, $n=5$ (place $n-1=4$ bars)

$$\underbrace{\circ\ \circ\ \circ}_{x_1=3}\ \Big|\ \underbrace{\circ}_{\substack{x_2\\ \shortparallel \\ 1}}\ \Big|\underbrace{\ }_{\substack{x_3\\ \shortparallel \\ 0}}\Big|\ \underbrace{\circ\ \circ\ \circ}_{x_4=3}\ \Big|\ \underbrace{\circ}_{x_5=1}$$



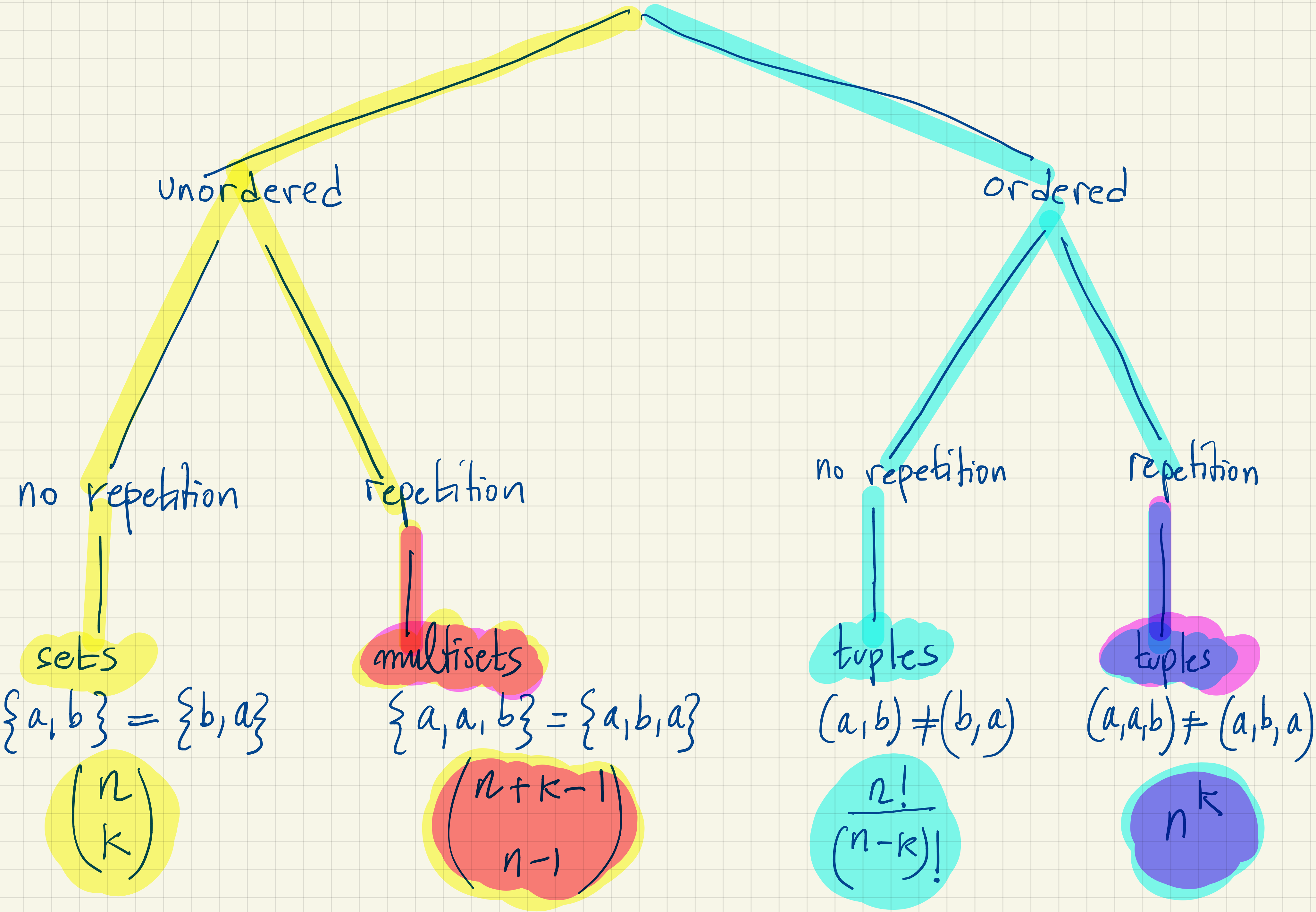
A binary word with $n+k-1$ bits and $n-1$ 1s

$$\binom{n+k-1}{n-1} = \binom{\#\text{ bits}}{\#\text{ 1s}}$$

Select k out of n

	no order	order
no repetition :	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$	$k!\binom{n}{k}$
repetition :	$\binom{n+k-1}{n-1}$	n^k

Today



unordered
ordered
no repetition
repetition
no repetition
repetition
sets
$\{a, b\} = \{b, a\}$
$\binom{n}{k}$
multisets
$\{a, a, b\} = \{a, b, a\}$
$\binom{n+k-1}{n-1}$
tuples
$(a, b) \neq (b, a)$
$\frac{n!}{(n-k)!}$
tuples
$(a, a, b) \neq (a, b, a)$
n^k