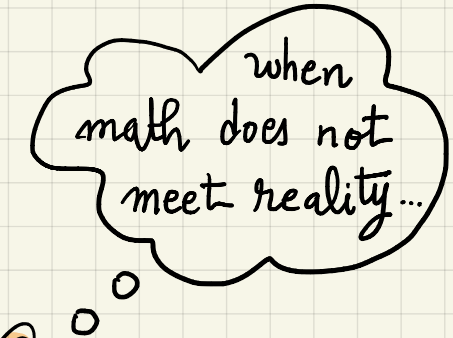
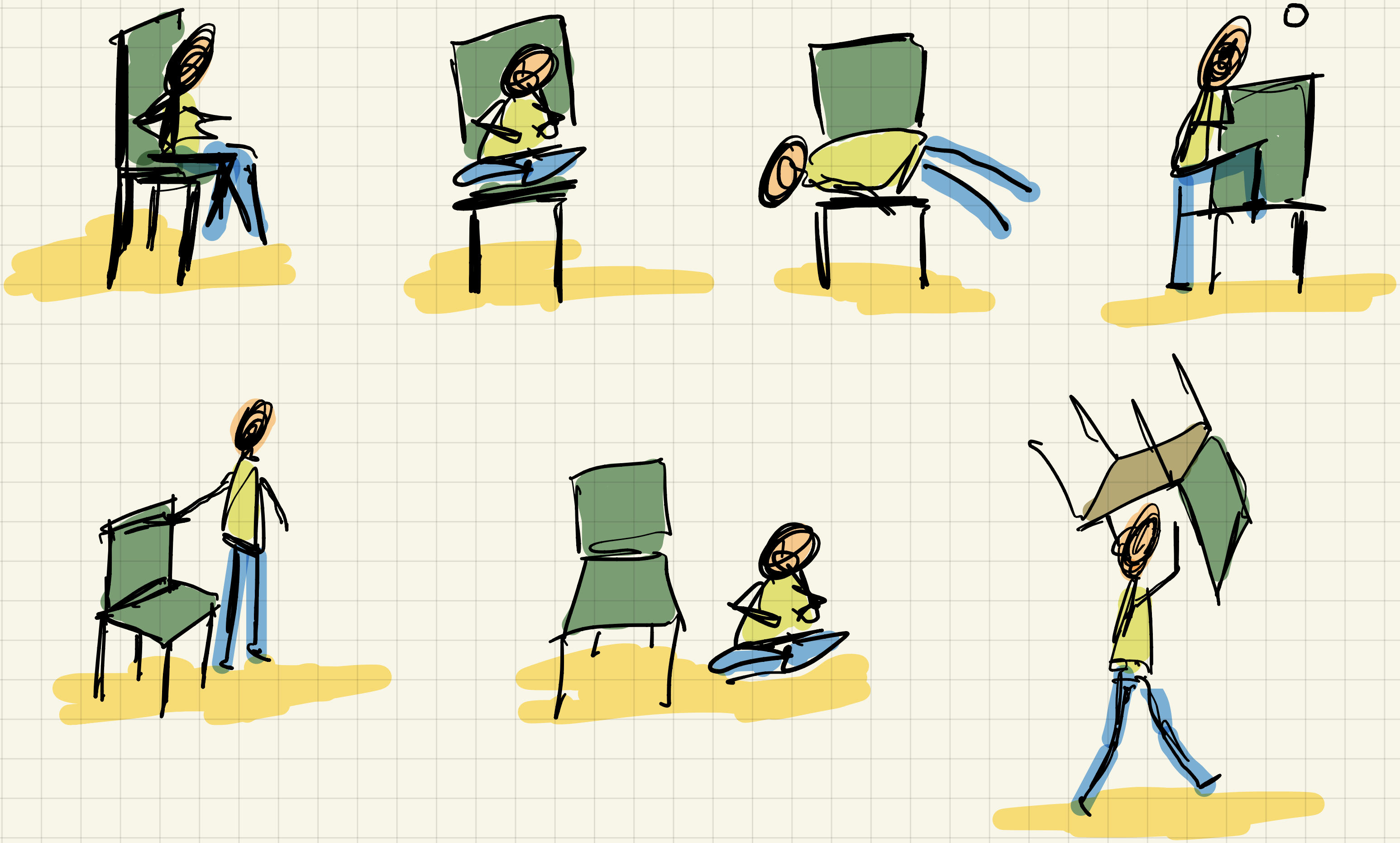
In how many ways can we seat one person on one chair?





lecture 8

We learned that the number of integer solutions to

$$x_1 + x_2 + \dots + x_n = k$$

$$x_i \geq 0$$

is $\dbinom{n+k-1}{n-1}$

Example: How many integer solutions are there for

$$x_1 + x_2 + x_3 = 15 \qquad x_i \geq 0$$

here $n = 3$, $k = 15$ so

$$\binom{3+15-1}{3-1} = \binom{17}{2} = \frac{17 \times 16}{2} = 136$$

What if ?

$$x_1 + x_2 + x_3 = 15$$

$$x_1 \geqslant 0$$

$$x_2 \geqslant 3$$

$$x_3 \geqslant 0$$

$x_2 \geqslant 3 \implies x_2 = 3 + y_2$ where $y_2 \geqslant 0$

$$x_1 + (3 + y_2) + x_3 = 15$$

$$x_1 + y_2 + x_3 = 12$$

$$x_1 \geqslant 0 \quad y_2 \geqslant 0 \quad x_3 \geqslant 0$$

Solve as before : $\binom{3+12-1}{3-1} = \binom{14}{2}$

Application:

Recall:
- # of binary words with n bits 2^n
- # of binary words with n bits and k 1's

  $\binom{n}{k}$

- # of binary words with n bits and k 1's and no consecutive 1's ?

Example: How many binary words have 10 bits, 3 1's, and no consecutive 1's ?

e.g. 0010001010

The 3 1's divide the 0's into 4 groups

$$\underbrace{\text{zeros}}_{x_1}\ 1\ \underbrace{\text{zeros}}_{x_2}\ 1\ \underbrace{\text{zeros}}_{x_3}\ 1\ \underbrace{\text{zeros}}_{x_4}$$

$$x_1 + x_2 + x_3 + x_4 = 7$$

$x_1 \geqslant 0$ $\quad$ $x_2 \geqslant 1$ $\quad$ $x_3 \geqslant 1$ $\quad$ $x_4 \geqslant 0$

let $\quad$ let $x_2 = 1 + y_2$ and $x_3 = 1 + y_3$ $\quad$ $y_i \geqslant 0$

$$x_1 + y_2 + y_3 + x_4 = 5$$

$x_1 \geqslant 0$ $\quad$ $y_2 \geqslant 0$ $\quad$ $y_3 \geqslant 0$ $\quad$ $x_4 \geqslant 0$

answer: $$\binom{4+5-1}{4-1} = \binom{8}{3}$$

Why $\binom{8}{3}$? There is another way to think about the problem by considering all 0's

$$. \ 0 \ . \ 0 \ . \ 0 \ . \ 0 \ . \ 0 \ . \ 0 \ . \ 0 \ .$$

The 7 0's define 8 positions as show above. Each position can take at most one 1. To place the 3 1's, we have to choose 3 positions out of 8. So $\binom{8}{3}$.

The Binomial Coefficients

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$ are called the **Binomial Coefficients**

(we will see why)

Some properties:

$$\binom{n}{k} = \binom{n}{n-k}$$

[symmetry]

example: $\binom{5}{3} = \binom{5}{2}$ why?

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$0 < k < n$

[Pascal Triangle]

ex: $\binom{5}{3} = \binom{4}{2} + \binom{4}{3}$

The Pascal Triangle

row 0 $\quad 1\binom{0}{0}$

row 1 $\quad \binom{1}{0}1 \quad 1\binom{1}{1}$

row 2 $\quad \binom{2}{0}1 \quad 2\binom{2}{1} \quad 1\binom{2}{2}$

row 3 $\quad \binom{3}{0}1 \quad \binom{3}{1}3 \quad \binom{3}{2}3 \quad \binom{3}{3}1$

$1 \quad 4 \quad 6\binom{4}{2} \quad 4\binom{4}{3} \quad 1$

$1 \quad 5 \quad 10 \quad 10\binom{5}{3} \quad 5 \quad 1$

$+1 \quad -6 \quad +15 \quad -20 \quad 15 \quad 6 \quad 1$

$\vdots$

$$(x+y)^0 = 1$$

$$(x+y)^1 = 1x + 1y$$

$$(x+y)^2 = 1.x^2 + 2xy + 1y^2$$

$$(x+y)^3 = 1.x^3 + 3x^2y + 3xy^2 + 1.y^3$$

$\vdots$

$(x+y)^n$ = binomial

polynomial of 2 variables

$$(x+y)^3 = (x+y)(x+y)(x+y)$$

$$\underbrace{(x^2+xy+yx+y^2)}(x+y)$$

$$(x^2+2xy+y^2)(x+y)$$

$$x^3+\underline{x^2y}+\underline{2x^2y}+2xy^2+xy^2+y^3$$

$$x^3+3x^2y+3xy^2+y^3$$

The Binomial theorem:

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \dots \binom{n}{n} x^0 y^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Proof: $\underbrace{(x+y)(x+y) \dots (x+y)}_{n} = \dots \boxed{?} x^{n-k} y^k \dots$

To generate $x^{n-k}y^k$ we have to pick k of the n factors to contribute y, that can be done in $\binom{n}{k}$ ways.

Example: $(x+y)^3 = \dots \; xy^2 \; \dots$

$$(x+y)(x+y)(x+y)$$

$$(x+y)(x+y)(x+y)$$

$$(x+y)(x+y)(x+y)$$

There are $\binom{3}{2}$ ways of contributing the y^2

In $(x+y)^n$, all terms are of the form $x^a y^b$

where $a+b=n$, so $(b=n-a)$

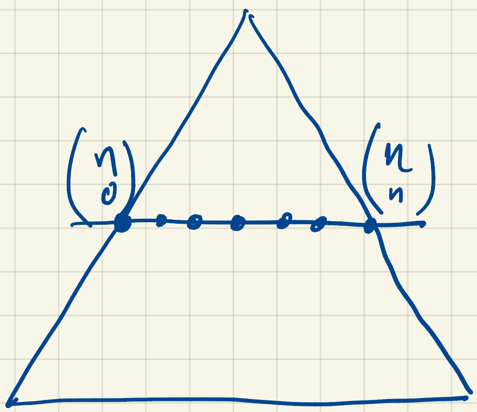
$$\binom{n}{a} = \binom{n}{b} \qquad \text{[Symmetry]}$$

Examples:

$$(1+1)^n = \binom{n}{0}1^n 1^0 + \binom{n}{1}1^{n-1}1^1 + \binom{n}{2}1^{n-2}1^2 + \cdots \binom{n}{n}1^0 1^n$$

$$= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots \binom{n}{n} = 2^n$$

n^{th} row



sum of binomial coefficients of row n is 2^n.

(Familiar quantity?) $\binom{n}{k}$ = # subsets of size k

so $\sum_{k=0}^{n}\binom{n}{k}$ = total # subsets (addition rule) $= 2^n$

$$(1-1)^n = \binom{n}{0}1^n(-1)^0 + \binom{n}{1}1^{n-1}(-1)^1 + \binom{n}{2}1^{n-2}(-1)^2 + \cdots \binom{n}{n}1^0(-1)^n$$

$$\big(\underbrace{1}_{x}+\underbrace{(-1)}_{y}\big)^n$$

$$(x+y)^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n\binom{n}{n} = 0^n$$

When $n>0$, the answer is 0.

So $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$

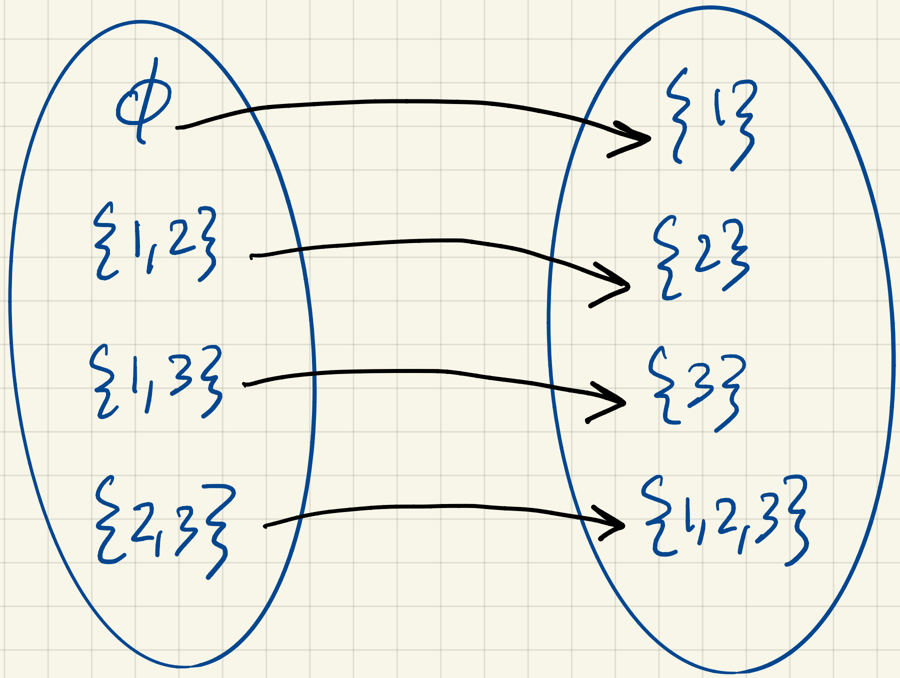
\# even subsets = \# odd subsets

<u>Example:</u> $n=3$ $S=\{1,2,3\}$

$\emptyset$ $\{1,2\}$ $\{1\}$ $\{2\}$

$\{1,3\}$ $\{2,3\}$ $\{3\}$ $\{1,2,3\}$

Another proof by bijection



$E(S)$

Even subsets of S

$O(S)$

odd subsets of S

$$f : E(S) \to O(S)$$

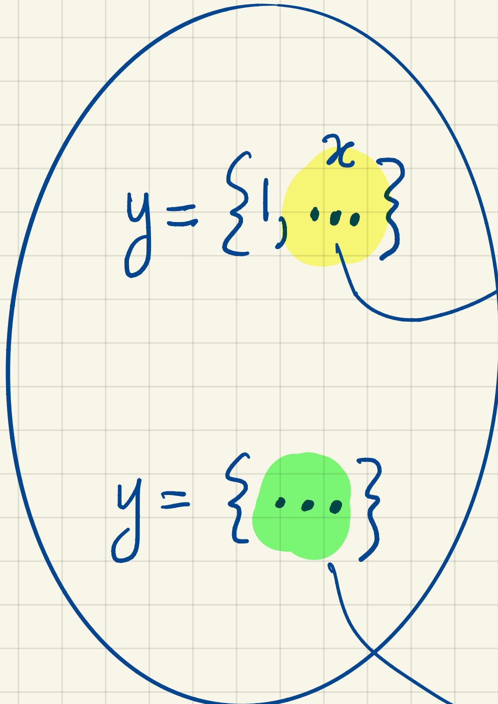
x, y are subsets of S

$x \in E(S)$

$y \in O(S)$

$$y = f(x) = \begin{cases} x - \{1\} & 1 \in x \\ x \cup \{1\} & 1 \notin x \end{cases}$$

onto:



$\mathcal{P}(S)$

this set x is even and has no 1

$\exists\, x \in E(S)$ such that $y = f(x)$

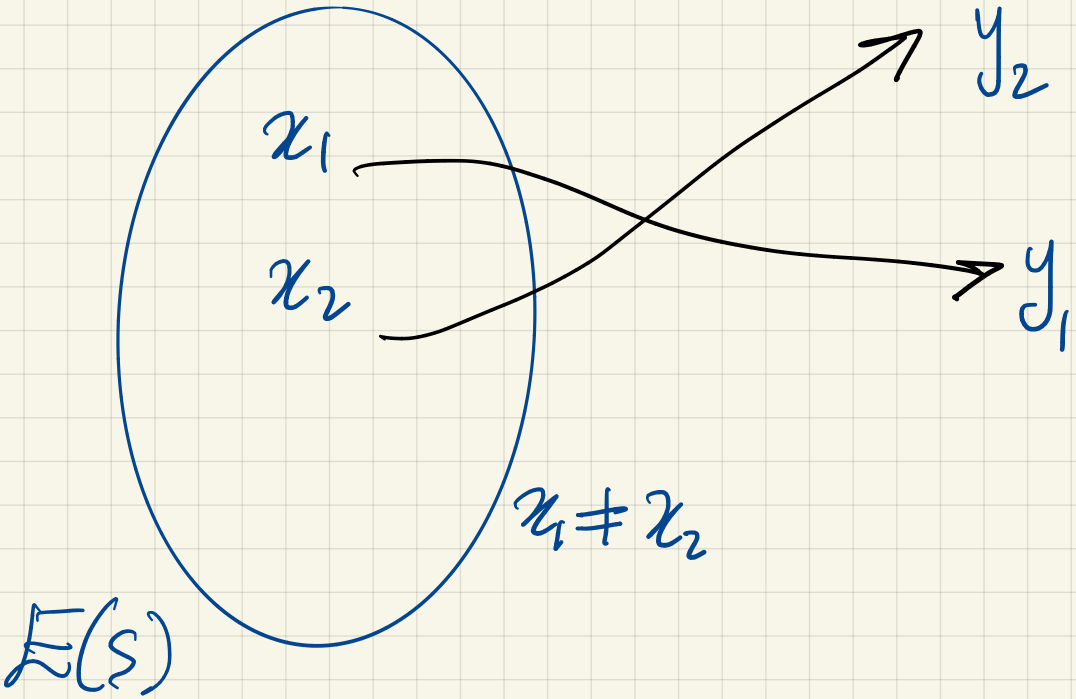
$x = y - \{1\}$

The set $y \cup \{1\}$ is even and has a 1 so

$\exists\, x \in E(S)$ such that $y = f(x)$

$x = y \cup \{1\}$

one-to-one:



$1 \in x_1$ and $1 \notin x_2 \implies 1 \notin y_1$ and $1 \in y_2 \implies y_1 \neq y_2$

$1 \in x_1$ and $1 \in x_2 \implies$ Removing the 1's makes $y_1 \neq y_2$

$1 \notin x_1$ and $1 \notin x_2 \implies$ adding 1's still makes $y_1 \neq y_2$