CSCI 150 Discrete Mathematics Test 1

Saad Mneimneh, Computer Science, Hunter College of CUNY

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Name: -

Recitation instructor (circle one): Adam Brandon

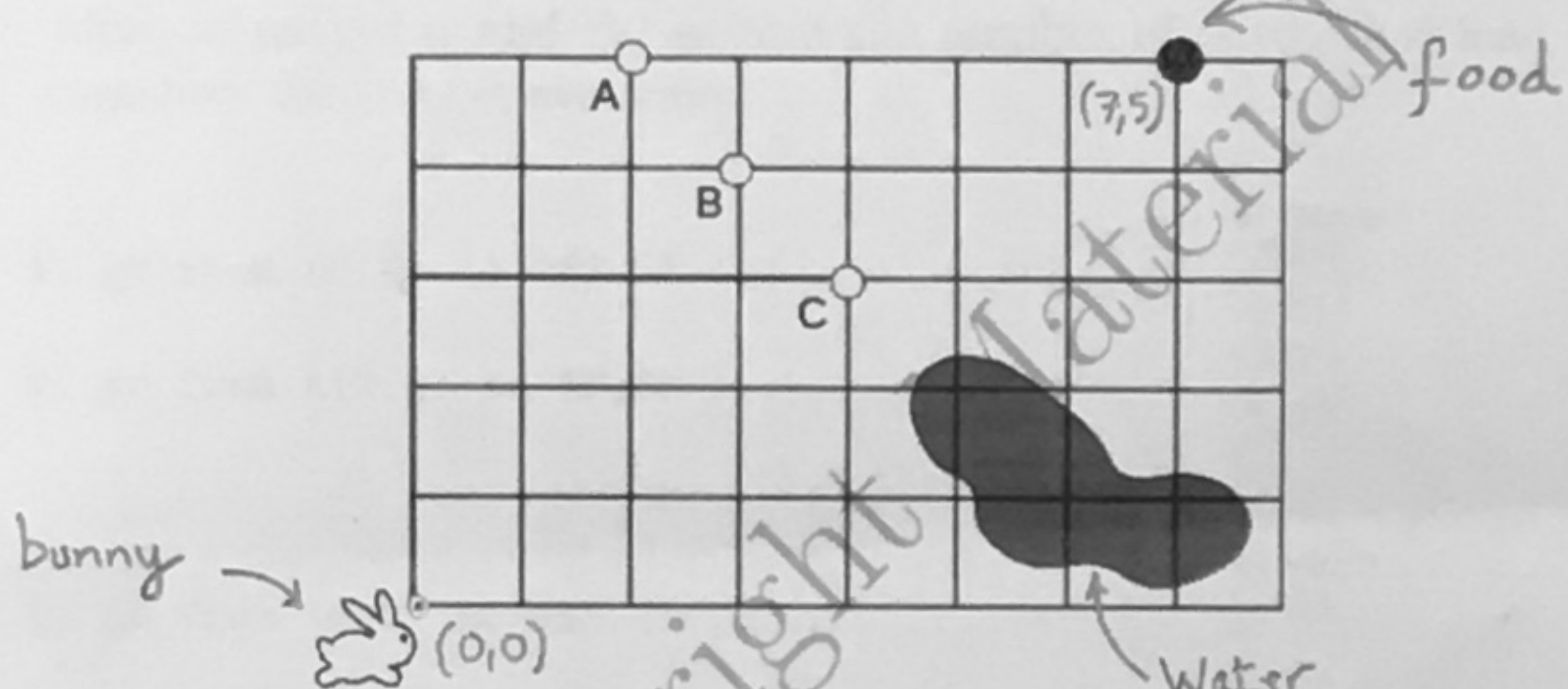
Recitation section (circle one):

`Mon 11:30` `Mon 12:30` `Mon 3:10` `Mon 4:00` `Wed 3:10`

`Wed 4:00` `Thu 11:30` `Thu 12:30` `Thu 3:10` `Thu 4:00`

Problem 1: The bunny who can't swim

A bunny stands at $(0, 0)$ with food located at $(7, 5)$ as shown below:



As it is typical in CSCI 150, the bunny can only make **UP** and **RIGHT** moves; however, this bunny can't swim and, therefore, will not be able to cross the body of water. The big question is: How many paths lead the bunny to his food? We will tackle this in several parts.

(a) (**2 points**) The bunny realized that he can divide the paths into three categories that are disjoint: the paths that pass through point $A(2, 5)$, the paths that pass through point $B(3, 4)$, and the paths that pass through point $C(4, 3)$. Describe in your own words the significance of such realization and how it relates to an important principle in counting.

Since the three categories are disjoint, the total number of paths is the sum of the numbers of paths in each category. That's the <u>Addition Rule</u>.

(b) (**2 points**) For any given point X, the bunny also realized that in order to go from $(0,0)$ to $(7,5)$ while passing through X, he must first go from $(0,0)$ to X, and then from X to $(7,5)$. Describe in your own words the significance of such realization and how it relates to an important principle in counting.

The number of ways to go from (0,0) to (7,5) passing through X is the product of two things.

- number of ways to go from (0,0) to X
- number of ways to go from X to (7,5)

This illustrates the <u>product rule</u> where the task is accomplished in two phases.

(c) (**4 points**) Use the ideas in parts (a) and (b) to find the number of paths that lead the bunny to his food. Consider the following worksheet for your convenience:

	# ways
1. go from (0,0) to A(2,5)	$\binom{7}{2}$
2. go from A(2,5) to (7,5)	$\binom{5}{0}$

	# ways
1. go from (0,0) to B(3,4)	$\binom{7}{3}$
2. go from B(3,4) to (7,5)	$\binom{5}{1}$

	# ways
1. go from (0,0) to C(4,3)	$\binom{7}{4}$
2. go from C(4,3) to (7,5)	$\binom{5}{2}$

Note 1: As we did in homework 4, a path from one point to another that makes only UP and RIGHT moves can be considered as an anagram of a given number of Us and Rs.

Note 2: For partial credit, if we assume that the numbers in the above worksheet are a, b, c, d, e and f (all of which can be expressed as binomial coefficients), give your answer in terms of these numbers.

The total number of paths is

$$\binom{7}{2}\binom{5}{0} + \binom{7}{3}\binom{5}{1} + \binom{7}{4}\binom{5}{2}$$

$$= \frac{7\times 6}{2}\times 1 + \frac{7\times 6\times 5}{3!}\times 5 + \frac{7\times 6\times 5}{3!}\times 10$$

$$= 21 + 35\times 5 + 35\times 10 = 546.$$

Problem 2: Sweet and sour and colorful
You have 13 sweet candies that taste the same and 7 sour candies that taste the same, but all 20 candies have different colors. Please answer the following counting questions and include a brief explanation for your reasoning.

(a) (**1 point**) In how many ways can you eat the candies if what matters is the sequence of colors? ANSWER:

$20!$ It's a permutation

(b) (**1 point**) In how many ways can you eat the candies if what matters is the sequence of flavors? *Hint*: think of anagrams. ANSWER:

$$\frac{20!}{13!\,7!} = \binom{20}{13} = \binom{20}{7}$$

Anagram with 13 0s and 7 1s where

Sweet $\equiv 0$

Sour $\equiv 1$

(c) (**1 point**) In how many ways can you eat the candies if you can't eat two sour candies in a row and what matters is the sequence of flavors? *Hint*: think of sweet as 0 and sour as 1. ANSWER:

~1~1~1~1~1~1~1~

The seven 1s divide the 0s into 8 parts

$$x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8 = 13$$

$x_1 \geq 0$ $\quad x_2, x_3, x_4, x_5, x_6, x_7 \geq 1$

$x_8 \geq 0$

we get $x_1 + x_2' + x_3' + x_4' + x_5' + x_6' + x_7' + x_8 = 7$

$x_i \geq 0$

\# solutions: $\binom{8+7-1}{8-1} = \binom{14}{7}$

<u>Another way</u>: $\cdot 0 \cdot 0 \cdot 0 \cdot 0 \cdot 0 \cdot 0 \cdot 0 \cdot 0 \cdot 0 \cdot 0 \cdot 0 \cdot 0 \cdot 0 \cdot$

choose 7 out of 14 positions for the 1s.

(d) (**1 point**) In how many ways can you eat only 5 candies if what matters is the sequence of flavors? *Hint:* you are selecting from {sweet, sour}. ANSWER:

2^5 (order & repetition of flavors)

(e) (**1 point**) In how many ways can you eat only 5 candies if what matters is the sequence of colors? ANSWER:

$$\frac{20!}{(20-5)!} = \frac{20!}{15!} = 16\times17\times18\times19\times20$$

(order & no repetition of colors)

(f) (**1 point**) In how many ways can you eat only 5 candies if what matters is how many sweet and how many sour candies you eat (but not their sequence)? ANSWER:

$n=2$ {sweet, sour}
$k=5$

Select 5

$$\binom{2+5-1}{2-1} = \binom{6}{1} = 6$$

(no order with repetition of flavors)

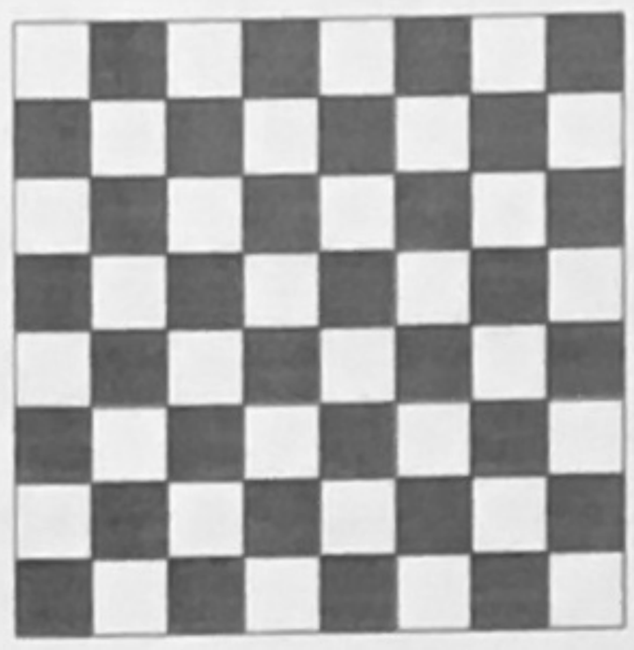
(g) (**1 point**) In how many ways can you eat only 5 candies if what matters is what colors you eat, but not their sequence? ANSWER:

$$\binom{20}{5} = \frac{20!}{15!\,5!}$$

Select 5 colors out of 20
(no order, no repetition)

Problem 3: Snakes and Ladders

Assume that you are playing the game of snakes and ladders on an 8×8 chessboard as shown below.



In addition, this version of the game is unique in that:

- There is only one snake
- There is only one ladder
- The head of the snake must be at a higher numbered square than its tail
- The head and tail of the snake must occupy different colors

(a) **(4 points)** Use the following procedure with the product rule to count the number of ways one can place the snake and the ladder (observe that for the product rule to work we must start with the snake). Don't worry about overcounting for now, just apply the product rule.

		# ways
snake	1. choose a black square	32
	2. choose a white square	32
ladder	3. choose another square	62
	4. choose another square	61
		$32 \times 32 \times 62 \times 61$

(b) (**2 points**) Adjust for overcounting and explain your reasoning.

The only overcount happens in the choices for the ladder. If we switch our choices for phases 3 & 4 we end up with the same outcome. Observe that snake is not overcounted because we choose black first then white, so we cannot permute our choices for phases 1 and 2.
Overall overcount is therefore by 2.

Answer: $\dfrac{32 \times 32 \times 62 \times 61}{2} = 1{,}936{,}384$

(c) (**2 points**) Consider the following function:

$$f : S \to S$$

where $S = \{1, 2, 3, \ldots, 64\}$ and

$$f(i) = \begin{cases} j & \text{if } i \text{ is the head of a snake and } j \text{ its tail} \\ j & \text{if } i \text{ is the bottom of a ladder and } j \text{ its top} \\ i & \text{otherwise} \end{cases}$$

Is f one-to-one? Is it onto? Explain your answers; for instance, if you answer YES provide a convincing argument, and if you answer NO, give a counter example.

It's not one-to-one: if i is the head of a snake and j its tail, then

$f(i) = f(j) = j$

It's not onto: if j is the head of a snake, there is no i such that $f(i) = j$.

The only way for f to be one-to-one/onto is if there is no snake or ladder on the board.

(d) (**2 points**) Assume now that you can make your own chessboard. A brand new 8×8 board came with all its 64 squares colored red. You can choose any number k, $0 \leq k \leq 64$, of squares, and then you can make each one either white or black. How many chessboards can you create? Find the number of chessboards for a given k first (in terms of k), and then use the addition rule.

Note: For full credit, first write your answer in Σ notation, then use the Binomial theorem to find a simplified form for your answer.

For a given k, there are $\binom{64}{k}$ ways of choosing the k squares, then 2^k ways of coloring them either black or white

By the addition rule:

$$\sum_{k=0}^{64} \binom{64}{k} 2^k$$

and by the Binomial theorem

$$\sum_{k=0}^{64} \binom{64}{k} 2^k = (1+2)^{64} = 3^{64}$$