CSCI 150 Discrete Mathematics
Test 1

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Name: Solution EmplID:

Recitation instructor (circle one): Morgan Brandon Enxhi
Recitation section (circle one):

`Mon 1:10` `Mon 2:10` `Mon 3:10` `Mon 4:00` `Wed 12:10`

`Wed 1:10` `Thu 1:10` `Thu 2:10` `Thu 3:10` `Thu 4:00`

There are 9 pages (including this one)

There are 4 problems (with several parts)

Don't turn the page until it's time

Scrap paper will be provided

Turn all your cell phones off and place them away (and no calculators)

If you need to leave (e.g. bathroom break), please give me all your cell phones

There is a total of 50 points, but the test will be graded over 40
(so you can get up to 10 extra points)

Problem 1: Quick answer questions (but maybe not so quick)
These questions are independent, solving a question does not rely on having solved another, the questions can be answered in any order. Each is worth 2 points.

(a) Consider the set $S = \{13, 19, 25, 31, \ldots, 601\}$. What is $|S|$?

$$\frac{601-13}{6} = 98$$

$$|S| = 99$$

(b) For the set S above, find the sum of all its elements.

Each element is $13+6i$, $i=0\ldots98$ (99 elements)

$$\sum_{i=0}^{98}(13+6i) = 99\times13 + 6\sum_{i=0}^{98} i = 99\times13 + 6\times\frac{98\times99}{2} = 30390$$

(c) A lottery ticket consists of 6 numbers in the set $\{1, 2, 3, \ldots, 49\}$. How many tickets are possible that don't have consecutive numbers?

Chosen number $\rightarrow 1$

Non chosen number $\rightarrow 0$

We have 43 zeros that define 44 positions, 1's go in these positions with at most one 1 in each. $\binom{44}{6}$

(d) You are given 10 books. In how many ways can you stack them?

$10!$ (permutations)

(e) For the same 10 books above, if you are lazy and you decide to stack only 5 of them (and leave the rest for another day), in how many ways can you do that?

$$\frac{10!}{(10-5)!} \quad (\text{permutations})$$

(f) If 10 books consist of 4 math books, 3 physics books, 2 chemistry books, and 1 biology book, in how many ways can you stack them if you only care about the sequence of topics?

$$\frac{10!}{4!3!2!1!}$$

(g) The alphabet is $\{a, b, c, \ldots, z\}$. How many 8 letter words can you make if a word must start with an 'a' and end with a 'z'?

$$a\underbrace{\ldots\ldots}_{6}z$$

$$26^6$$

(h) The alphabet is $\{a, b, c, \ldots, z\}$. How many 8 letter words are "double words"? A double word means that each of its letters appears exactly twice *Hint:* first choose which 4 letters are to be used, then think of anagrams.

$$\binom{26}{4} \times \frac{8!}{2!2!2!2!}$$

Problem 2: An interesting identity

(a) (1 point) Let $S = \{1, 2, 3, 4, 5, 6\}$. How many subsets of S have 3 elements? $\binom{6}{3}$

(b) (2 point) How many subsets of S have 3 elements if:

- 1 is the smallest element of the subset: $\binom{5}{2}$
- 2 is the smallest element of the subset: $\binom{4}{2}$
- 3 is the smallest element of the subset: $\binom{3}{2}$
- 4 is the smallest element of the subset: $\binom{2}{2}$

(c) (3 points) Explain how and why the addition rule can relate the two parts (a) and (b). Show the work.

$$\binom{5}{2} + \binom{4}{2} + \binom{3}{2} + \binom{2}{2} = \binom{6}{3}$$

We have disjoint categories of subsets of size 3.
The sum must be equal the total (addition rule)

(d) (6 points) In this part, you will try to generalize the work you did above. We will take S to be the set $\{1, 2, 3, \ldots, n\}$.

- (1 pt) How many subsets of S have k elements? $\binom{n}{k}$

- (2 pts) How many subsets of S have k elements if i is the smallest among them? $\binom{n-i}{k-1}$

- (3 pts) Find an interesting identity involving a sum of binomial coefficients by using the addition rule. Your identity must have the form:

$$\sum_{i=\ldots}^{\ldots} \binom{\ldots}{\ldots} = \binom{\ldots}{\ldots}$$

$$\sum_{i=1}^{n-k+1} \binom{n-i}{k-1} = \binom{n}{k}$$

Note: $n-k+1$ can be changed to n if we observe that $\binom{a}{b} = 0$ when $a<b$.

(e) (2 points) Take $n = 6$ and $k = 3$ and illustrate the identity you obtained in part (d) on the Pascal triangle.

1

1 1

1 2 + 1

1 3 + 3 1

1 4 + 6 4 1

1 5 10 10 5 1

=

1 6 15 20 15 6 1

$$\binom{5}{2} + \binom{4}{2} + \binom{3}{2} + \binom{2}{2} = \binom{6}{3}$$

Problem 3: Chickens

We have 20 chickens in a row, where each chicken must be oriented either to the left or to the right. An example orientation is shown below:

1 2 3 ... 20

(a) (2 points) How many possible orientations are there? 2^n

Define the visibility of chicken i as the number of chickens it sees. For instance, chicken 1 above sees 19 chickens, and chicken 3 sees 2 chickens. Therefore, consider the visibility function

$$f : \{1, 2, \ldots, 20\} \to \{0, 1, \ldots, 19\}$$

So we would say $f(1) = 19$ and $f(3) = 2$.

(b) (4 points) How many orientations make $f(1) \geq f(20)$? *Hint*: Use the following procedure to guide you.

```
                                                # ways
1.  choose orientations for chicken 1 and chicken 20    3
2.  choose an orientation for chicken 2                 2
3.  choose an orientation for chicken 3                 2
. . .                                                   ⋮
19. choose an orientation for chicken 19                2
                                                 ---------------
```

3×2^{18}

(c) (4 points) How many orientations make f an onto function?

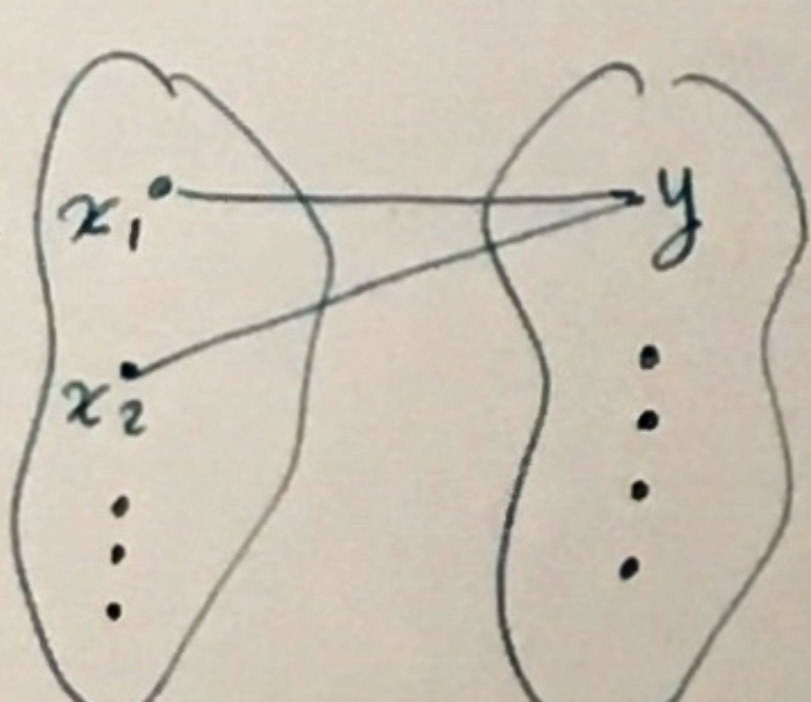
Hint: Think which chickens can have visibility 0 and 19 and how the two chickens must be oriented. Once 0 and 19 are achieved, what about other visibility values?

Chicken 1 and 20 must be oriented in the same way to get either $f(1)=0$ and $f(20)=19$ or $f(1)=19$ and $f(20)=0$.
So this can be done in two ways.
Following that, the same argument applies for chicken 2 and chicken 19 to get visibility 1 and 18, etc...
So we have $\underbrace{2 \times 2 \times \dots \times 2}_{10} = 2^{10}$ ways.

(d) (2 points) Explain in your own words why in this particular setting, if f is onto, then it's also one-to-one.

Since the domain and the codomain have the same size (20 in this case), any onto function will also be one-to-one (and vice versa).

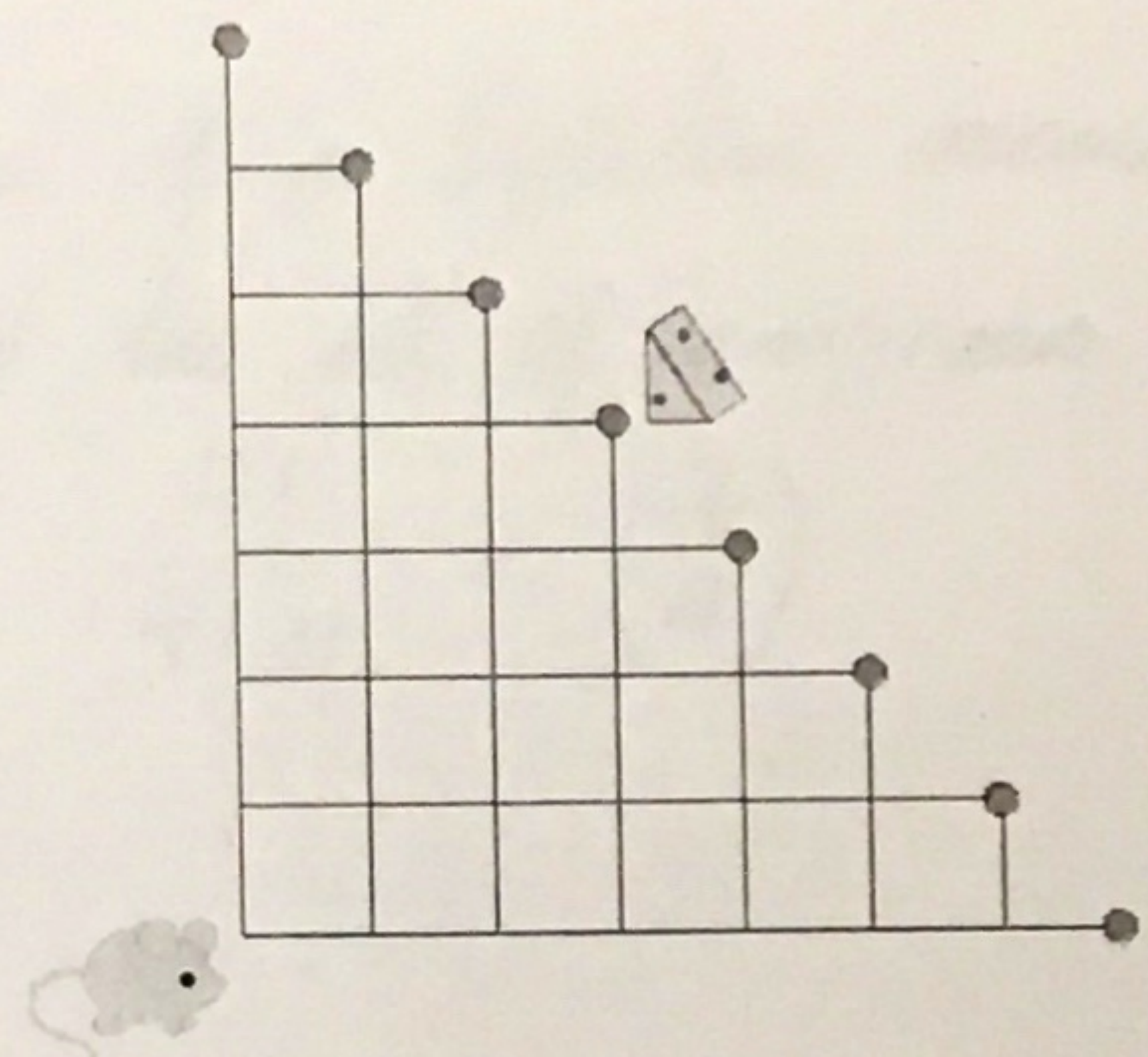
Illustration:



if not one-to-one, there will be not enough x's to map to all y's, and therefore not onto.

Problem 4: The mouse and the cheese

A mouse sits at $(0,0)$ and some positions are shown with a circle. One of the circles has a cheese as shown below.



The mouse can make Right and Up moves on the grid. A path is a sequence of moves that bring the mouse to the cheese.

(a) (2 points) Describe in clear terms a bijection from the set of paths to the set of anagrams of the word

URURURU

Given a path, we transform it into an anagram by replacing every Right move with R and every Up move with U. There will always be 7 moves, 4 of which are Us and 3 of which are Rs.

(b) (2 points) Explain the significance of the existence of such bijection, and hence find the number of ways the mouse can get to the cheese.

The existence of a bijection means that the set of paths and the set of anagrams have the same size, thus $\frac{7!}{4!3!} = \binom{7}{4}$

(c) (4 points) If all the circles in the above figure have a cheese and, in addition, each Up move can be either a *walk*, a *run*, or a *hop*, in how many ways can the mouse get to a cheese?

Hint: Come up with an expression that will make use of the Binomial theorem.

There are $\binom{7}{k}$ path with k Ups. Each of these paths ~~be~~ can be realized in 3^k ways.

Summing over all possible k, we have

$$\sum_{k=0}^{7} \binom{7}{k} 3^k = (3+1)^7 = 4^7 \text{ ways}$$

by the Binomial theorem.