CSCI 150 Discrete Mathematics
Homework 10

Saad Mneimneh, Computer Science, Hunter College of CUNY

Due Mon. 5/6/2024 (before midnight)

1. Find the greatest common divisor of 100 and 254 using prime factorization. What is their least common multiple?
2. Find the greatest common divisor of 100 and 254 using the Euclidean algorithm and express it as a linear combination of 100 and 254 like this: $\gcd(254, 100) = 254r - 100s$, where $r, s \geq 0$.
3. Let F_n be the nth Fibonacci number. What is $\gcd(F_{2024}, F_{2023})$?
4. What do the following pairs of integers have in common: two consecutive numbers, two consecutive odd numbers, two consecutive Fibonacci numbers, two prime numbers, a prime number p and an integer a such that $p \not| a$, and a prime number p and an integer $a < p$?
5. Show that if $a|bc$ and $\gcd(a, b) = 1$, then $a|c$.
6. Given $n > 1$, let p be a prime number such that $n < p < 2n$ (by the way, there is always a prime between n and $2n$). Does p divide $\binom{2n}{n}$? Explain.
7. Imagine you have points on a circle labeled $0, 1, 2, \ldots, 126$, so point 126 is followed by point 0. You start at point 0, and you repeatedly jump by 5, so you first land on point 5, then 10, then 15, etc... How many jumps do you need to land on point 1? Try to think about the mathematical concept needed to figure this out without guessing.

8. For each of these relations, specify whether it is reflective, symmetric, anti-symmetric, and transitive.

   - The subset relation on the power set of some set S
   - The relation $\leq$ on $\mathbb{R}$
   - The relation $<$ on $\mathbb{Z}$
   - The relation "shared a class with" on the set of students at Hunter College, where two students share a class if there is a class they are both enrolled in this semester.
   - The relation given by

     $$\{(a,c),(a,f),(a,h),(b,h),(c,f),(c,h),(d,h),(e,h),(f,h),(g,h)\}$$

   - The relation R on $\mathbb{N}$ where $(a,b) \in R$ means $a|b$
   - The relation R on $\mathbb{N}$ where $(x,y) \in R$ means $x < y+2$

9. Consider the following relation on $\mathbb{N} \times \mathbb{N}$ (the set of ordered pairs of positive integers):

   $$(a,b) \equiv (c,d) \Leftrightarrow ab = cd$$

   Prove that this is an equivalence relation and prove that for any integer $n \in \mathbb{N}$, there exist classes of equivalence of size n. *Hint*: Think about why this is the same as saying that there exist integers that have n divisors.

   Are there finitely many or infinitely many classes of equivalence of size 1? of size 2?

10. Every non-empty subset of $\mathbb{N}$ (whether it's finite or infinite) has a minimum. One can't say the same about $\mathbb{Z}$. Find a total order relation $\prec$ on $\mathbb{Z}$ such that every non-empty subset of $\mathbb{Z}$ has a minimum under the $\prec$ relation.

Not all the questions above will be graded, but work on all of them to practice the concepts.

The following are optional and will not be graded (for the curious).

- Consider the number 60, it is factored into primes as $2^2 \cdot 3 \cdot 5$. Observe now that $(2^0+2^1+2^2)(3^0+3^1)(5^0+5^1)$ gives the sum of all the divisors of 60 due to the distributive law. Find the sum of all the divisors of 1000.

- Find the product of all the divisors of 1000. This requires a little bit of thinking along the lines of the previous question.

- Fermat's theorem says that if p is prime and $a < p$, then

$$a^{p-1} \equiv 1 \pmod{p}$$

  What is $2^{75} \mod 71$? *Note*: This is easier than the example in the recitation.

- Find positive $x < 17$ and $y < 17$ that satisfy:

$$2x + y \equiv 4 \pmod{17}$$
$$5x - 5y \equiv 9 \pmod{17}$$

  Note: This can be done by "pretending" that $\equiv$ is $=$ and solving as usual, except that when dividing by an integer, one must multiply by its inverse modulo 17. Refer to example in recitation.