CSCI 150 Discrete Mathematics
Homework 2
Due 10/2/09

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Problem 1: Pairs and mathematical identities
The number of ways to pair $2n$ players in n teams of two is

$$\frac{(2n)!}{2^n n!}$$

The above expression can be obtained by ordering $2n$ players $-$ in $(2n)!$ ways $-$ and then accounting for the fact that many orders correspond to the same pairing. So we adjust for this overcounting by dividing $(2n)!$ by 2^n and $n!$.

(a) Explain why we overcount by $2^n n!$.

(b) Obtain an expression for the number of ways to pair $2n$ players using a different counting technique: start with a pool of $2n$ players, and repeatedly (actually n times) choose 2 players from the pool to make a pair. Show that this process results in the following expression (use the notion of phases and compute the number of ways each of the n phases can be carried out):

$$\binom{2n}{2} \cdot \binom{2n-2}{2} \cdot \ldots \cdot \binom{2}{2}$$

How much overcounting is done using this process? Explain clearly. *Hint*: see next question.

(c) Show that

$$\binom{2n}{2} \cdot \binom{2n-2}{2} \cdot \ldots \cdot \binom{2}{2} = \frac{(2n)!}{2^n}$$

using

- a combinatorial argument based on the above counting
- algebraically

(d) Yet another way to count is the following: Perform n phases where, in each phase, let the youngest unpaired player choose a partner. Obtain an expression based on this counting strategy and argue that no overcounting is done. Then prove the following identity using a combinatorial argument:

$$(2n-1)\cdot(2n-3)\cdot\ldots\cdot 1 = \frac{(2n)!}{2^n n!}$$

Problem 2: One ball two ball red ball blue ball...
Find the number of ways of placing 4 balls in 10 distinguishable bins if:

(a) the balls are distinguishable, and no bin can hold more than one ball.

(b) the balls are indistinguishable, and no bin can hold more than one ball.

(c) the balls are distinguishable, and each bin can hold any number of them.

(d) the balls are indistinguishable, and each bin can hold any number of them.

Problem 3: Sets
Show that for any three sets A, B, and C:

$$((A\backslash B)\cup(B\backslash A))\cap C = ((A\cap C)\cup(B\cap C))\backslash(A\cap B\cap C)$$

(a) by letting $S = ((A\backslash B)\cup(B\backslash A))\cap C$ and $T = ((A\cap C)\cup(B\cap C))\backslash(A\cap B\cap C)$ and showing that

- $x \in S \Rightarrow x \in T$, *Hint*: given an element a and two sets X and Y, $a \notin X \Rightarrow a \notin X \cap Y$.
- $x \in T \Rightarrow x \in S$, *Hint*: given an element a and two sets X and Y, $a \in X$ and $a \notin X \cap Y \Rightarrow a \in X\backslash Y$

(b) using Venn diagrams

Problem 4: Playing with mathematics
How many different words (not necessarily found in the dictionary) can we get by rearranging the letters in the word MATHEMATICS?

Problem 5: Where is my seat?
In how many ways can you seat n people on a round table with n chairs if:

(a) each chair has a different color

(b) all chairs are identical, but each person cares about his/her neighbors

(c) all chairs are identical, each person cares about his/her neighbors, but not on which side they sit

(d) if each person cares about who is sitting on his/her left and right

Problem 6: Yet another property of binomial coefficients
Show that

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

(a) algebraically

(b) by providing a combinatorial argument. *Hint*: think of the expression on the right as a counting process consisting of two stages (a process that actually overcounts).