CSCI 150 Discrete Mathematics
Homework 2
Solution

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Problem 1: Pairs and mathematical identities
The number of ways to pair $2n$ players in n teams of two is

$$\frac{(2n)!}{2^n n!}$$

The above expression can be obtained by ordering $2n$ players − in $(2n)!$ ways − and then accounting for the fact that many orders correspond to the same pairing. So we adjust for this overcounting by dividing $(2n)!$ by 2^n and $n!$.

(a) Explain why we overcount by $2^n n!$.

Solution: We can imagine that each team corresponds to two empty spots. Forming the teams would be to assign the players to the spots. This is essentially the same as imposing some order on the players, which can be done in $(2n)!$ ways. However, by doing so, we are actually overcounting, for the following reasons: A permutation of teams results in a different assignment, however, does not change the outome. This is a factor of $n!$. Moreover, switching the two players in each team corresponds to a different assignment, however, does not change the outcome. This is a factor of 2 for each team, leading to 2^n.

(b) Obtain an expression for the number of ways to pair $2n$ players using a different counting technique: start with a pool of $2n$ players, and repeatedly (actually n times) choose 2 players from the pool to make a pair. Show that this process results in the following expression (use the notion of phases and compute the number of ways each of the n phases can be carried out):

$$\binom{2n}{2} \cdot \binom{2n-2}{2} \cdot \ldots \cdot \binom{2}{2}$$

How much overcounting is done using this process? Explain clearly. *Hint*: see next question.

Solution: We can think of this process as being carried out in n phases. In the first phase, we have to choose 2 players from a pool of 2n players. Therefore, the first phase can be carried out in $\binom{2n}{2}$ ways. Having done the first phase, in the second phase we have $2n-2$ players. Therefore, the second phase can be done in $\left(\begin{array}{c}2n-2\\2\end{array}\right)$ ways. This is repeated until the last phase, when we are left with 2 players only, and the last phase can be carried out in $\left(\begin{array}{c}2\\2\end{array}\right)$ ways. The entire process can be then carried out in a number of ways corresponding to the multiplication of all these terms, as indicated above. By doing so, we are overcounting by $n!$, because permuting the choices made in each phase does not change the outcome.

(c) Show that

$$\left(\begin{array}{c}2n\\2\end{array}\right)\cdot\left(\begin{array}{c}2n-2\\2\end{array}\right)\cdot\ldots\cdot\left(\begin{array}{c}2\\2\end{array}\right)=\frac{(2n)!}{2^n}$$

using

- a combinatorial argument based on the above counting
- algebraically

Solution: Using a combinatorial argument, we showed in (b) that

$$\left(\begin{array}{c}2n\\2\end{array}\right)\cdot\left(\begin{array}{c}2n-2\\2\end{array}\right)\cdot\ldots\cdot\left(\begin{array}{c}2\\2\end{array}\right)$$

overcounts the number of possible pairs by a factor if $n!$. Therefore,

$$\frac{\left(\begin{array}{c}2n\\2\end{array}\right)\cdot\left(\begin{array}{c}2n-2\\2\end{array}\right)\cdot\ldots\cdot\left(\begin{array}{c}2\\2\end{array}\right)}{n!}=\frac{(2n)!}{2^n n!}$$

which immediately implies the result if we multiply each side by $n!$.

Algebraically:

$$\left(\begin{array}{c}2n\\2\end{array}\right)\cdot\left(\begin{array}{c}2n-2\\2\end{array}\right)\cdot\ldots\cdot\left(\begin{array}{c}2\\2\end{array}\right)=$$

$$\frac{(2n)!}{2!(2n-2)!}\cdot\frac{(2n-2)!}{2!(2n-4)!}\cdot\frac{(2n-4)!}{2!(2n-6)!}\cdot\ldots\cdot\frac{2!}{2!0!}$$

By simplifying identical numerators and denominators, we end up with $\frac{(2n!)}{2^n}$.

(d) Yet another way to count is the following: Perform n phases where, in each phase, let the youngest unpaired player choose a partner. Obtain an expression based on this counting strategy and argue that no overcounting is done. Then prove the following identity using a combinatorial argument:

$$(2n-1)\cdot(2n-3)\cdot\ldots\cdot 1=\frac{(2n)!}{2^n n!}$$

Solution: In the first phase, the yongest player has a choice of $2n-1$ partners. Once this choice is made, the yongest of the remaining $2n-2$ players, has a choice of $2n-3$ partners. This is repeated until two players are left, and the yonger among the two has only one choice to make. The entire process can then be carried out as indicated on the left side of the expression above. The interesting thing about this approach is that it does not overcount. Given a paricular outome, there is only one way that this outcome could have been obtained: the yongest must have made the first choice, so we can always determine what the first choice was. Once this is done, the remaining yongest must have made the second choice, etc... Therefore, using a combinatorial argument, we can establish that:

$$(2n-1)\cdot(2n-3)\cdot\ldots\cdot 1 = \frac{(2n)!}{2^n n!}$$

Problem 2: One ball two ball red ball blue ball...
Find the number of ways of placing 4 balls in 10 distinguishable bins if:

(a) the balls are distinguishable, and no bin can hold more than one ball.

Solution: In this case we have to select a bin for each ball; therefore, we select $k = 4$ out of $n = 10$ bins with order. That's

$$\frac{n!}{(n-k)!} = \frac{10!}{6!} = 10 \cdot 9 \cdot 8 \cdot 7 = 5040$$

(b) the balls are indistinguishable, and no bin can hold more than one ball.

Solution: Same as above, except that order is not important. So it is just a matter of selecting $k = 4$ out of $n = 10$ bins. That's

$$\frac{n!}{k!(n-k)!} = \frac{10!}{4!6!} = 210$$

(c) the balls are distinguishable, and each bin can hold any number of them.

Solution: Now each bin can be selected more than once. So we need to select $k = 4$ out of $n = 10$ with repetition. Order is important because the balls are distinguishable (think: balls are gifts, bins are kids). That's

$$n^k = 10^4 = 10000$$

(d) the balls are indistinguishable, and each bin can hold any number of them.

Solution: Same as above, except that order is not important anymore (think: balls are pennies, bins are kids). The answer is:

$$\binom{n+k-1}{n-1} = \binom{13}{9} = 715$$

Problem 3: Sets
Show that for any three sets A, B, and C:

$$((A\backslash B) \cup (B\backslash A)) \cap C = ((A \cap C) \cup (B \cap C))\backslash(A \cap B \cap C)$$

(a) by letting $S = ((A\backslash B)\cup(B\backslash A))\cap C$ and $T = ((A\cap C)\cup(B\cap C))\backslash(A\cap B\cap C)$ and showing that

- $x \in S \Rightarrow x \in T$, *Hint*: given an element a and two sets X and Y, $a \notin X \Rightarrow a \notin X \cap Y$.
- $x \in T \Rightarrow x \in S$, *Hint*: given an element a and two sets X and Y, $a \in X$ and $a \notin X \cap Y \Rightarrow a \in X\backslash Y$

Solution:
$\underline{((A\backslash B) \cup (B\backslash A)) \cap C \subseteq ((A \cap C) \cup (B \cap C))\backslash(A \cap B \cap C)}$

$x \in ((A\backslash B) \cup (B\backslash A)) \cap C) \Rightarrow x \in (A\backslash B) \cup (B\backslash A)$ and $x \in C \Rightarrow$
$(x \in A\backslash B$ or $x \in B\backslash A)$ and $x \in C \Rightarrow$
$(x \in A\backslash B$ and $x \in C)$ or $(c \in B\backslash A$ and $x \in C) \Rightarrow$
$(x \in A$ and $x \notin B$ and $x \in C)$ or $(x \in B$ and $x \notin A$ and $x \in C) \Rightarrow$
$(x \in A \cap C$ and $x \notin B)$ or $(x \in B \cap C$ and $x \notin A) \Rightarrow$
$(x \in A \cap C$ and $x \notin B \cap A \cap C)$ or $(x \in B \cap C$ and $x \notin A \cap B \cap C) \Rightarrow$
$(x \in A \cap C$ or $x \in B \cap C)$ and $x \notin A \cap B \cap C \Rightarrow$
$x \in (A \cap C) \cup (B \cap C)$ and $x \notin A \cap B \cap C \Rightarrow$
$x \in ((A \cap C) \cup (B \cap C))\backslash(A \cap B \cap C)$

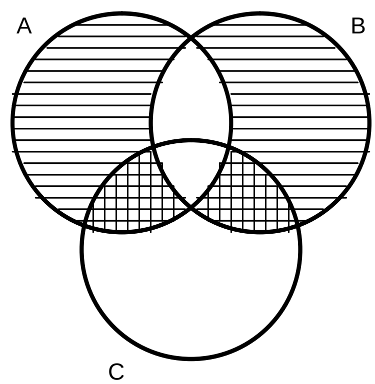
$\underline{((A \cap C) \cup (B \cap C))\backslash(A \cap B \cap C) \subseteq ((A\backslash B) \cup (B\backslash A)) \cap C}$

We can reverse the direction of the arrows in the above proof.

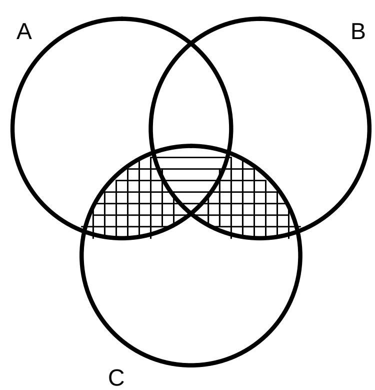
(b) using Venn diagrams

Solution:

$$((A\backslash B)\cup(B\backslash A))\cap C$$



$$((A\cap C)\cup(B\cap C))\backslash(A\cap B\cap C)$$



Problem 4: Playing with mathematics
How many different words (not necessarily found in the dictionary) can we get by rearranging the letters in the word MATHEMATICS?

Solution: We have 11 letters and, therefore, 11! possible permutations of these letters. However, some of these permutations are identical, because permuting similar letters does not change the word. For instance, permuting the "M"s in MATHEMATICS preserves the word. Therefore, the word MATHEMATICS can be obtained in 2! ways by simply permuting the "M"s among themselves. But we can also permute other similar letters. "M" appears 2 times, "A" appears 2 times, "T" appears 2 times, "H" appears 1 time, "E" appears 1 time, "I" appears 1 time, "C" appears 1 time, and "S" appears 1 time. Therefore, each word can be obtained in $2! \cdot 2! \cdot 2! \cdot 1! \cdot 1! \cdot 1! \cdot 1! \cdot 1!$ ways. The number of distinct words is

$$\frac{11!}{2! \cdot 2! \cdot 2! \cdot 1! \cdot 1! \cdot 1! \cdot 1! \cdot 1!} = \frac{11!}{8}$$

Problem 5: Where is my seat?
In how many ways can you seat n people on a round table with n chairs if:

(a) each chair has a different color

Solution: This is identical to ordering the people, so the number of ways is $n!$.

(b) all chairs are identical, each person cares about his/her neighbors, but not on which side they sit

Solution: If we seat the people in $n!$ ways, then a seating remains the same when rotated (chairs are identical). Therefore, the number is $n!/n = (n-1)!$. Another way of thinking about this is to seat the first person on a given chair (that choice is always fixed), then seat the remaining $(n-1)$ people on the remaining $(n-1)$ chairs. Every seating is different now.
(d) all chairs are identical, each person cares about who is sitting on his/her left and right

Solution: As before, except that now a seating remains the same when either rotated or inverted. The number of ways we can get the same seating is now $2n$. Therefore, we have $n!/(2n) = (n-1)!/2$ ways. Another way of thinking about this is to seat the first person on a given seat (fixed choice), and choose his/her two neighbors in $\binom{n-1}{2}$ ways. Then, repeatedly choose one person from the remaining $n-3$ to complete a missing neighbor of a person already sitting. Therefore,

$$\binom{n-1}{2}(n-3)\ldots 1 = \frac{(n-1)!}{2}$$

Note that the second step of choosing 2 neighbors for the first person is crutial to avoid overcounting (why?).

Problem 6: Yet another property of binomial coefficients
Show that

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

(a) algebraically

Solution:

$$k\binom{n}{k} = k\frac{n!}{k!(n-k)!} = k\frac{n(n-1)!}{k(k-1)!(n-k)!} = n\frac{(n-1)!}{(k-1)!(n-k)!}$$

$$= n\frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} = n\binom{n-1}{k-1}$$

(b) by providing a combinatorial argument. *Hint*: think of the expression on the right as a counting process consisting of two stages (a process that actually overcounts).

Solution: We can choose k from n by first choosing one element (n possibilities) and then choosing $k-1$ from the remaining $n-1$ elements ($\left(\begin{array}{c} n-1 \\ k-1 \end{array} \right)$ possibilities). But by doing so, we get every subset of k elements exactly k times, depending on which of the its elements was chosen first. So that's $k \left(\begin{array}{c} n \\ k \end{array} \right)$.