CSCI 150 Discrete Mathematics
Homework 3
Solution

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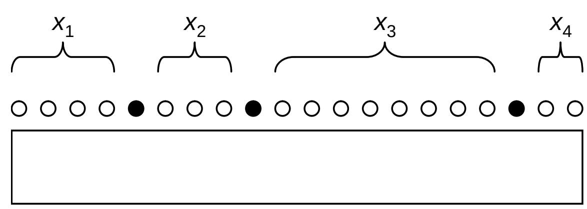
Problem 1: Neighbors

(a) Consider 20 persons sitting on one side of a rectangular table. How many ways can we choose 3 persons, no two of whom are neighbors?

Hint: the choice of three persons partitions the table into 4 sets of people S_1, S_2, S_3, and S_4, where $|S_1| + |S_2| + |S_3| + |S_4| = 17$ and $S_2 \neq \emptyset$ and $S_3 \neq \emptyset$.

Solution: Consider a choice of 3 persons. This choice partitions the table into 4 groups from left to right with x_1, x_2, x_3, and x_4 persons respectively, such that $x_1 + x_2 + x_3 + x_4 = 17$.



Moreover, since no two of the 3 persons are neighbors, $x_2 \geq 1$ and $x_3 \geq 1$ (x_1 and x_4 may be zero, e.g. choosing a person at the extremity of the table). Therefore, $x_2 = x_2' + 1$ and $x_3 = x_3' + 1$, and we have:

$$x_1 + x_2' + x_3' + x_4 = 15$$

Choosing 3 persons no two of whom are neighbors gives a solution to the above equation. Similarly, a solution to the above equation implies a specific choice of 3 persons, no two of who are neighbors. Therefore, there is a one to one correspondence between the two sets, and the number of solutions to the above equation is equal to the number of choices that we can make.
This is equivalent to distributing 15 pennies on 4 children. Therefore, the answer is:

$$\binom{15+4-1}{15} = \binom{18}{15} = \frac{18!}{15!3!} = 816$$

(b) What if the 20 persons are sitting on a round table?

Hint: thinking about a partitioning of the table into three sets S_1, S_2, and S_3, such that $|S_1|+|S_2|+|S_3| = 17$ and non of the sets is empty does not work. Why?

Another Hint: select a person to be the head of the table (this defeats the purpose of a round table, but it serves the math). If this person is not chosen, then we have to choose three persons on a rectangular table with 19 persons. If on the other hand that person is chosen, then we have to choose two persons on a rectangular table with 17 persons.

Solution: One may argue, in a similar way to part (a), that the choice of three persons no two of whom are neighbors partitions the **round** table into three groups with x_1, x_2, and x_3 persons respectively, such that:

$$x_1 + x_2 + x_3 = 17$$

$$x_1 \geq 1, x_2 \geq 1, x_3 \geq 1$$

Solving this as before will give

$$\binom{14+3-1}{14} = 120$$

However, this reasoning is wrong. For the round table, we do not have a one to one correspondence between the number of choices and the number of solutions to the above equation. For instance, say $x_1 = 4$, $x_2 = 5$, and $x_3 = 8$. Many choices of 3 persons give such partitioning of the round table (because the table is round).

Another way to approach this problem is by breaking the round table into a rectangular one. For instance, designate a person P to be the head of the table (will defeat the purpose of a round table but will serve the math). P may or may not be among the 3 chosen ones. If P is among the chosen ones, then both of its neighbors cannot be chosen, and it remains to choose 2 persons that are not neighbors on a rectangular table of 17 people. If P is not among the chosen ones, then we must choose 3 persons no two of whom are neighbors on a rectangular table of 19 people. Solving the two problems separately will give:

$$\binom{14+3-1}{14} + \binom{14+4-1}{14} = 120 + 680 = 800$$

Another way is the following: we observe that any choice on the round table is valid on the rectangular table. But a choice on the rectangular table is not necessarily valid on the round table, namely, a choice that selects the two persons at the two extremities of the table (these are neighbors on the round table). But we only have 16 such choices on the rectangular table (16 choices for the third person). Therefore, the number of valid choices on the round table is 816-16=800.

Yet another way: We have 20 choices for the first person. Then we have 17 choices for the second person. Two of these 17 choices (the ones to the left and right of the first choice) will give 15 choices for the third person, and 15 of these 17 choices (the rest) will give 14 choices for the third person. By the principle of phases we have $20(2 \times 15 + 15 \times 14) = 4800$. But we are overcounting because any permutation of the first, second, and third person gives the same choice. Therefore, we must divide by 3! to get:

$$\frac{20(2 \times 15 + 15 \times 14)}{3!} = 800$$

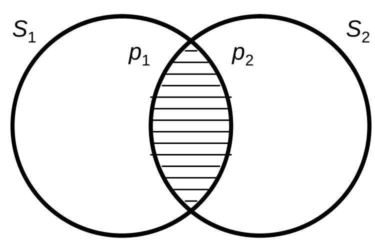
Problem 2: A party

Consider a party of 60 persons. Assume that each one knows at least m others.

(a) How large must m be to guarantee that 3 persons know each other?

Solution: It is not hard to see that m must be greater than 30. For if we consider the scenario of two groups of 30 people each, where knowledge is only across groups, then each person knows 30 others, and yet we cannot find three people that know each other. We can show that $m > 30$ is enough.

Assume $m > 30$. Let p_1 be a person and define S_1 to be the set of people p_1 knows. Then $|S_1| > 30$. Now pick a person $p_2 \in S_1$, and define S_2 to be the set of people p_2 knows. Then $|S_2| > 30$ also. This is illustrated below using Venn diagrams.



We would like to show that $S_1 \cap S_2$ is not empty, i.e. $|S_1 \cap S_2| > 0$. From the inclusion-exclusion principle on sets S_1 and S_2:

$$|S_1 \cap S_2| = |S_1| + |S_1| - |S_1 \cup S_2| > 30 + 30 - 60 = 0$$

The inequality follows because $|S_1| > 30$, $|S_2| > 30$, and $|S_1 \cup S_2| \leq 60$.

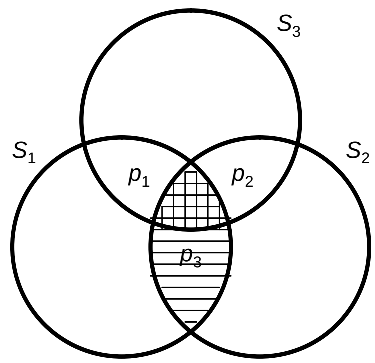
(b) How large must m be to guarantee that 4 persons know each other?

Solution: It is not hard to see that m must be greater than 40. For if we consider the scenario of three groups of 20 people each, where knowledge is only across groups, then each person knows 40 others, and yet we cannot find four people that know each other. We can show that $m > 40$ is enough.

Assume $m > 40$. Let p_1 be a person and define S_1 to be the set of people p_1 knows. Then $|S_1| > 40$. Now pick a person $p_2 \in S_1$, and define S_2 to be the set of people p_2 knows. Then $|S_2| > 40$ also. As we did for part (a):

$$|S_1 \cap S_2| = |S_1| + |S_1| - |S_1 \cup S_2| > 40 + 40 - 60 = 20$$

Therefore $|S_1 \cap S_2| > 20$. Now pick a person $p_3 \in S_1 \cap S_2$, and define S_3 to be the set of people p_3 knows. Then $|S_3| > 40$. This is illustrated below using Venn diagrams.



We would like to show that $S_1 \cap S_2 \cap S_3$ is not empty, i.e. $|S_1 \cap S_2 \cap S_3| > 0$. From the inclusion-exclusion principle on sets $S_1 \cap S2$ and S_3:

$$|(S_1 \cap S_2) \cap S_3| = |S_1 \cap S_2| + |S_3| - |(S_1 \cap S_2) \cup S_3| > 20 + 40 - 60 = 0$$

The inequality follows because $|S_1 \cap S_2| > 20$, $|S_3| > 40$, and $|(S_1 \cap S_2) \cup S_3| \leq 60$.

Hint: Use the inclusion exclusion principle, as illustrated in class.

Problem 3: Beware of Pokemon
There is a contest with 40 Pokemons. There are 18 Pokemons who like to fight in the sky, and 23 who like to fight on ground. Several of them like to fight in water. The number of those who like to fight in the sky and on ground is 9. There are 7 Pokemons who like to fight in the sky and in water, and 12 who like to fight on ground and in water. There are 4 Pokemons who like to fight in the sky, on ground, and in water. How many Pokemons like to fight in water?

Solution: Let S, G, and W be the sets of Pokemons who like to fight in the sky, on the ground, and in water, respectively. Then $|S| = 18$, $|G| = 23$, $|S \cap G| = 9$, $|S \cap W| = 7$, $|G \cap W| = 12$, $|S \cap G \cap W| = 4$, and $|S \cup G \cup W| = 40$. By the inclusion-exclusion principle:

$$|S \cup G \cup W| = |S| + |G| + |W| - |S \cap G| - |S \cap W| - |G \cap W| + |S \cap G \cap W|$$

Therefore,

$$|W| = |S \cup G \cup W| - |S| - |G| + |S \cap G| + |S \cap W| + |G \cap W| - |S \cap G \cap W|$$

$$= 40 - 18 - 23 + 9 + 7 + 12 - 4 = 23$$

Problem 4: A great idea for Pigeonhole
We select 38 even positive integers, all less than 1000. Prove that there will be two of them whose difference is at most 26.

Solution: Partition the set $\{2, \ldots, 998\}$ into sets of size 27 as follows:

$$\{2,\ldots,998\} = \{2,\ldots 28\} \cup \{29,\ldots 55\} \cup \ldots \cup \{946,\ldots,972\} \cup \{973,\ldots 998\}$$

Note that the size of the last set is less than 27. More importantly, we have only 37 sets; therefore, 2 numbers must belong to the same set. As a result, these 2 numbers differ by at most 26.