

# CSCI 150 Discrete Mathematics

## Homework 4

### Solution

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#### Problem 1: The binary system

Show by induction that every natural number can be represented by a sum of **distinct** powers of 2.

(a) regular induction: *Hint*: when you assume the property is true for  $n - 1$ , come up with an expression for  $n - 1$ . You will find that it is a sum of powers of 2 with different coefficients (of course you don't know what the coefficients are). So

$$n - 1 = \sum_{i=0}^k a_i 2^i$$

Consider the first coefficient that is zero. Use this information and the fact that  $\sum_{i=0}^{j-1} 2^i = 2^j - 1$ , which we proved in class, to make the inductive step work.

**Solution:** We can prove by induction that any natural number can be written as a sum of **distinct** powers of 2.

**Base case:**  $n = 0$  can be represented as the empty sum, so the property is trivially true for  $n = 0$ . We may start with  $n = 1$  if the concept of an empty sum is troubling.  $n = 1 = 2^0$ .

**Inductive step:** Assume the property is true for  $n - 1$ ; therefore,

$$n - 1 = a_0 \cdot 2^0 + a_1 \cdot 2^1 + \dots + a_m \cdot 2^m$$

where  $a_i \in \{0, 1\}$  for  $i = 0 \dots m$ . Let  $a_j$  be the first zero coefficient. Then

$$\begin{aligned} n - 1 &= 2^0 + 2^1 + \dots + 2^{j-1} + 0 \cdot 2^j + a_{j+1} \cdot 2^{j+1} + \dots + a_m \cdot 2^m \\ &= \sum_{i=0}^{j-1} 2^i + a_{j+1} \cdot 2^{j+1} + \dots + a_m \cdot 2^m \\ &= 2^j - 1 + a_{j+1} \cdot 2^{j+1} + \dots + a_m \cdot 2^m \end{aligned}$$

But  $n = (n - 1) + 1$ , so:

$$n = 2^j + a_{j+1} \cdot 2^{j+1} + \dots + a_m \cdot 2^m$$

Therefore,  $n$  can be written as the sum of distinct powers of 2.

(b) strong induction. *Hint:* let  $k$  be the largest integer such that  $n = m + 2^k$ . Use strong induction on  $m$  to obtain the result.

Here's another proof that uses strong induction:

**Base case:** same as before.

**Inductive step:** Assume that the property is true for  $0, 1, 2, \dots, n-1$ . Given  $n$ , let  $k$  be the largest integer such that  $n = m + 2^k$ . By the choice of  $k$ , we know that  $m < 2^k$ ; otherwise  $n \geq 2^k + 2^k = 2^{k+1}$ . Since  $m \in \{0, 1, 2, \dots, n-1\}$ ,  $m$  can be written as the sum of distinct powers of 2, non of which is  $2^k$ . Therefore, since  $n = m + 2^k$ ,  $n$  is also the sum of distinct powers of 2.

**Problem 2: What is that sum?**

(a)

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{(n-1) \cdot n}$$

**Solution:**

$$n = 2: \frac{1}{1 \cdot 2} = 1/2$$

$$n = 3: \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = 2/3$$

$$n = 4: \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} = 3/4$$

We conjecture that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{(n-1) \cdot n} = \frac{n-1}{n}$$

Let's prove it by induction.

**Base case:**  $n = 2$ ,  $\frac{n-1}{n} = 1/2$ .

**Inductive step:** Assume the property is true for  $n-1$ . Then

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{(n-2) \cdot (n-1)} = \frac{n-2}{n-1}$$

Now,

$$\begin{aligned} & \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{(n-1) \cdot n} \\ &= \left[ \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{(n-2) \cdot (n-1)} \right] + \frac{1}{(n-1) \cdot n} \\ &= \frac{n-2}{n-1} + \frac{1}{(n-1)n} = \frac{n(n-2) + 1}{(n-1)n} = \frac{n^2 - 2n + 1}{(n-1)n} = \frac{(n-1)^2}{(n-1)n} = \frac{n-1}{n} \end{aligned}$$

(b)

$$0 \cdot \binom{n}{0} + 1 \cdot \binom{n}{1} + 2 \cdot \binom{n}{2} + \dots + (n-1) \cdot \binom{n}{n-1} + n \cdot \binom{n}{n}$$

**Solution:**

$$n = 0: 0 \cdot \binom{0}{0} = 0$$

$$n = 1: 0 \cdot \binom{1}{0} + 1 \cdot \binom{1}{1} = 1$$

$$n = 2: 0 \cdot \binom{2}{0} + 1 \cdot \binom{2}{1} + 2 \cdot \binom{2}{2} = 4$$

$$n = 3: 0 \cdot \binom{3}{0} + 1 \cdot \binom{3}{1} + 2 \cdot \binom{3}{2} + 3 \cdot \binom{3}{3} = 12$$

$$n = 4: 0 \cdot \binom{4}{0} + 1 \cdot \binom{4}{1} + 2 \cdot \binom{4}{2} + 3 \cdot \binom{4}{3} + 4 \cdot \binom{4}{4} = 32$$

We observe that if we divide the result by  $n$ , we get the following sequence: 0, 1, 2, 4, 8, ... We conjecture that:

$$0 \cdot \binom{n}{0} + 1 \cdot \binom{n}{1} + 2 \cdot \binom{n}{2} + \dots + (n-1) \cdot \binom{n}{n-1} + n \cdot \binom{n}{n} = n2^{n-1}$$

Let's prove it by induction.

**Base case:**  $n = 0$ ,  $n2^{n-1} = 0$ .

**Inductive step:** Assume the property is true for  $n - 1$ , then

$$0 \cdot \binom{n-1}{0} + 1 \cdot \binom{n-1}{1} + \dots + (n-1) \cdot \binom{n-1}{n-1} = (n-1)2^{n-2}$$

Now consider the sum:

$$0 \cdot \binom{n}{0} + 1 \cdot \binom{n}{1} + \dots + (n-1) \cdot \binom{n}{n-1} + n \cdot \binom{n}{n}$$

We can use the identity  $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$  for  $0 < k < n$ , and the fact that  $\binom{n}{0} = \binom{n-1}{0}$  and  $\binom{n}{n} = \binom{n-1}{n-1}$  to expose the term  $(n-1)$ .

$$\begin{aligned} & 0 \cdot \binom{n}{0} + 1 \cdot \binom{n}{1} + \dots + (n-1) \cdot \binom{n}{n-1} + n \cdot \binom{n}{n} \\ &= 0 \cdot \binom{n-1}{0} + 1 \cdot \left[ \binom{n-1}{0} + \binom{n-1}{1} \right] + \dots \\ & \dots + (n-1) \left[ \binom{n-1}{n-2} + \binom{n-1}{n-1} \right] + n \cdot \binom{n-1}{n-1} \end{aligned}$$

Regrouping terms we get:

$$\begin{aligned} & 0 \cdot \binom{n-1}{0} + \dots + (n-1) \cdot \binom{n-1}{n-1} \\ & + 0 \cdot \binom{n-1}{0} + \dots + (n-1) \cdot \binom{n-1}{n-1} \end{aligned}$$

$$+ \binom{n-1}{0} + \dots + \binom{n-1}{n-1}$$

Using the inductive hypothesis and the binomial theorem we get:

$$\begin{aligned} & 0 \cdot \binom{n}{0} + 1 \cdot \binom{n}{1} + \dots + (n-1) \cdot \binom{n}{n-1} + n \cdot \binom{n}{n} \\ &= (n-1)2^{n-2} + (n-1)2^{n-2} + 2^{n-1} = (n-1)2^{n-1} + 2^{n-1} = n2^{n-1} \end{aligned}$$

Experiment, conjecture the value, and then prove it by induction. The second one is hard, so ask me.

### Problem 3: Some Fibonacci identities

Show the following:

(a)  $F_2 + F_4 + F_6 + \dots + F_{2n} = F_{2n+1} - 1$   
(hint: use induction)

#### Solution:

Base case:  $n = 1$ :  $F_2 = F_{2 \cdot 1 + 1} - 1 = F_3 - 1 = 2 - 1 = 1$ .

Inductive step: Assume  $F_2 + F_4 \dots + F_{2(n-1)} = F_{2(n-1)+1} - 1$ . We need to prove that  $F_2 + F_4 \dots + F_{2n} = F_{2n+1} - 1$ .

$$\begin{aligned} F_2 + F_4 \dots + F_{2n} &= [F_2 + \dots + F_{2(n-1)}] + F_{2n} = F_{2(n-1)+1} - 1 + F_{2n} \\ &= F_{2n-1} - 1 + F_{2n} = F_{2n-1} + F_{2n} - 1 = F_{2n+1} - 1 \end{aligned}$$

(b)  $F_{n+1}^2 - F_n^2 = F_{n-1}F_{n+2}$   
(hint: don't use induction)

**Solution:** Observe that  $a^2 - b^2 = (a-b)(a+b)$ . Then  $F_{n+1}^2 - F_n^2 = (F_{n+1} - F_n)(F_{n+1} + F_n) = F_{n-1}F_{n+2}$ .

(c)  $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$   
(hint: use  $F_{a+b+1} = F_{a+1}F_{b+1} + F_aF_b$  on  $a+b+1 = 2n-1$  for different values of  $a$  and  $b$ )

**Solution:** We will use induction and the shifting property of the Fibonacci sequence.

Base case:  $n = 1$ :  $F_0F_2 - F_1^2 = 0 \cdot 1 - 1^2 = (-1)^1$ .

Inductive step: Assume  $F_{n-2}F_n - F_{n-1}^2 = (-1)^{n-1}$ . We need to prove that  $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$ . For this consider the shifting property of Fibonacci.

$$F_{2n-1} = F_{(n-1)+(n-1)+1} = F_{n-1}F_{n-1} + F_nF_n$$

$$F_{2n-1} = F_{(n-2)+n+1} = F_{n-1}F_{n+1} + F_{n-2}F_n$$

Therefore,  $F_{n-1}F_{n+1} + F_{n-2}F_n = F_{n-1}F_{n-1} + F_nF_n$ . This implies that  $F_{n-1}F_{n+1} - F_n^2 = -[F_{n-2}F_n - F_{n-1}^2] = (-1) \cdot (-1)^{n-1} = (-1)^n$ .

**Problem 4: Recurrence**

Find an explicit value for

$$a_n = 3 \cdot a_{n-1} - 2 \cdot a_{n-2}$$

given that  $a_0 = 0$  and  $a_1 = 1$

**Solution:** We consider the equation  $x^2 = 3x - 2$ . This equation has two solutions  $p = 1$  and  $q = 2$ . Therefore,  $a_n$  can be written as:

$$a_n = c_1 \cdot 1^n + c_2 \cdot 2^n$$

Now we can solve for  $c_1$  and  $c_2$  based on our knowledge of  $a_0$  and  $a_1$ .

$$a_0 = c_1 \cdot 1^0 + c_2 \cdot 2^0 = c_1 + c_2 = 0$$

$$a_1 = c_1 \cdot 1^1 + c_2 \cdot 2^1 = c_1 + 2c_2 = 1$$

This means that  $c_1 = -1$  and  $c_2 = 1$ . Therefore,

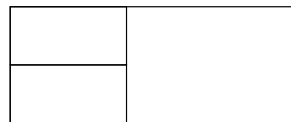
$$a_n = -1^n + 2^n = 2^n - 1$$

**Problem 5: Paving a garden path**

A path is 2 meters wide and  $n$  meters long. It is to be paved using paving stones of size  $1\text{m} \times 2\text{m}$ . In how many ways can the paving be accomplished?

*Hint:* Rabbits may cross the path.

**Solution:** Let  $P_n$  be the number of pavings for a  $2 \times n$  path. The paving must start with one of the options shown below:



In the first case it can be completed in  $P_{n-1}$  ways; in the second case it can be completed in  $P_{n-2}$  ways. So,

$$P_n = P_{n-1} + P_{n-2}$$

Clearly  $P_1 = 1$  and  $P_2 = 2$ . Therefore,  $P_n = F_{n+1}$ , where  $F_n$  is the  $n^{\text{th}}$  Fibonacci number. Therefore,

$$P_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1} \right]$$