Problem 1: Lattice points
We have five lattice points (points with integers as coordinates) in two-dimensional space. Show that the midpoint of one of the line segments that connect these points is also a lattice point. **Hint:** Each point has two coordinates each of which can be either even or odd.

**Solution:** A point can be categorized as (even, even), (even, odd), (odd, even), or (odd, odd) based on the parity of its two coordinates. If we imagine 4 boxes for these categories, placing the points in 4 boxes means that one of the boxes must contain at least two points. Let \( p = (x, y) \) and \( q = (a, b) \) be two such points. Then \( x \) and \( a \) have the same parity, and \( y \) and \( b \) have the same parity. This means \( x + a \) is even, and \( y + b \) is even. Therefore, the mid point
\[
z = \left( \frac{x + a}{2}, \frac{y + b}{2} \right)
\]
has integer coordinates.

Problem 2: Divisibility by \( n \)
Consider a list of \( n + 1 \) positive integers:
\[
a_1, a_2, \ldots, a_n, a_{n+1}
\]
(a) Prove that it is always possible to choose a pair of these whose difference is divisible by \( n \). **Hint:** For each \( a_i \), consider the remainder in the division by \( n \).

**Solution:** Consider the remainder in the division by \( n \). The remainder belongs to the set \( \{0, 1, 2, \ldots, n-1\} \). Each \( a_i \) is placed in the corresponding box based on its remainder in the division by \( n \). Since we have \( n \) boxes and \( n + 1 \) numbers, two of them must fall in the same box. Let \( a_i \) and \( a_j \) be those two. Then:
\[
a_i = nq_i + r
\]
\[
a_j = nq_j + r
\]
This means \( a_i - a_j = n(q_i - q_j) + (r - r) \), so a multiple of \( n \).

(b) Suppose \( a_{n+1} \) is now dropped from the list and \( n > 2 \). Prove that it is always possible to choose a pair whose sum or difference is divisible by \( n \). **Hint:**
What happens if you put $a_1, \ldots, a_n$ in $n$ boxes based on their remainder in the division by $n$?

**Solution:** We now only have $n$ numbers and $n$ boxes. We don’t have the guarantee that two numbers will fall in the same box anymore. But if two numbers fall in the same box, we are done (just consider their difference). If not, this means that every box contains exactly one element. Since $n > 2$, box 1 and box $n-1$ are distinct. Pick the corresponding elements, we have:

$$a_i = nq_i + 1$$
$$a_j = nq_j + (n-1)$$

The sum of these two is a multiple of $n$.

**Problem 3: Languages**

In a certain class there are 25 students: 14 speak Spanish, 12 speak French, 6 speak French and Spanish, 5 speak German and Spanish, and 2 speak all three. The 6 that speak German all speak another language. How many speak no foreign language?

**Solution:**

\[
\begin{align*}
|S| &= 14, \\
|F| &= 12, \\
|F \cap S| &= 6, \\
|G \cap S| &= 5, \\
|F \cap G \cap S| &= 2, \\
|G| &= 6
\end{align*}
\]

$|G \cap F|$ is missing but we can determine it from the information that says all German speaking students also speak another language. Therefore,

\[
G = (G \cap S) \cup (G \cap F)
\]

\[
|G| = |G \cap F| + |G \cap S| - |G \cap F \cap S|
\]

\[
6 = |G \cap F| + 5 - 2
\]

\[
|G \cap F| = 3
\]

Now,

\[
|G \cup F \cup S| = 14 + 12 + 6 - 6 - 5 - 3 + 2 = 20
\]

So 20 students speak a foreign language. Therefore, 5 students don’t.

Note: Another way to solve the problem is to say the following: since $G$ is entirely covered by other sets, i.e., no element in $G$ that is not in another set, then $G \cup F \cup S$ is the same as $F \cup S$, so we can do inclusion-exclusion on $F$ and $S$ alone to obtain the result.

**Problem 4: Card game**

You are playing cards with 3 other players, call them Player 1, Player 2, and Player 3. You draw a 10. You will lose if any player gets J, Q, K, or A. What
is the probability that you will lose? Hint: Say that a hand is good for Player $i$ if he gets J, Q, K, or A. Let $S_i$ be the set of hands that are good for Player $i$.

(a) Using Inclusion-Exclusion, find the number of good hands $|S_1 \cup S_2 \cup S_3|$. Dividing this by the total number of hands gives you the probability.

**Solution:** 51 cards remain and 16 cards are good.

$$|S_1| = |S_2| = |S_3| = 16 \times 50 \times 49$$
$$|S_1 \cap S_2| = |S_1 \cap S_3| = |S_2 \cap S_3| = 16 \times 15 \times 49$$
$$|S_1 \cap S_2 \cap S_3| = 16 \times 15 \times 14$$

Therefore,

$$|S_1 \cup S_2 \cup S_3| = 3(16 \times 50 \times 49) - 3(16 \times 15 \times 49) + 16 \times 15 \times 14 = 85680$$

The total number of hands is $51 \times 50 \times 49 = 124950$. Therefore, the probability is $85680/124950 = 0.6857$.

(b) Find the number of good hands in another way: First find the number of bad hands, then subtract it from the total number of hands.

**Solution:** The number of hands that are not good for any player is $35 \times 34 \times 33 = 39270$. Therefore, the number of good hands is $124950 - 39270 = 85680$, as before.