CSCI 150 Discrete Mathematics
Homework 6

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Due Wed. 3/20/2024 (before midnight)

We argued in class that there are no proofs by example, an exception being that we may disprove a statement by providing a *counter example*. For instance, to prove that the polynomial $p(n) = n^2 + n + 41$ does not produce a prime number for every integer $n \geq 0$, we may show that $41^2 + 41 + 41$ is not prime.

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1. Prove that the polynomial $n^2 - 79n + 1601$ does not produce a prime number for every integer $n \geq 0$.

2. Prove that the sequence given by

$$a_n = 1 + \prod_{k=1}^{n} p_k$$

where p_k is the k^{th} prime, is not always prime. Here are the first few values (which are prime):

$$3, 7, 31, 211, 2311, \ldots$$

Another scenario where the use of an example is appropriate is *existential proofs* when we are interested in showing the truth of a statement of the form:

$$\exists n, P(n)$$

For example, prove that there exist a prime number that is even.

$$\exists n, n \text{ is prime and even}$$

In this case, we can simply "construct" an example. For instance, 2 is prime and is even. Done!
Here's another example: Prove that there exists two perfect squares whose sum is a perfect square.

$$\exists x, y, z \in \mathbb{N}, x^2 + y^2 = z^2$$

Similarly, we can construct an example: $9 + 16 = 25$.

3. The Fibonacci numbers are given by the following sequence:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

   Prove that there is a Fibonacci number that ends in the digit 7.

4. Prove that there exists two irrational numbers x and y such that xy is rational.

Sometimes, it is not easy to construct an explicit example, but we can still prove existence. Such proofs are called "non-constructive". Here's an example: Prove that $x^3 + x - 1 = 0$ has a solution.

$$\exists x \in \mathbb{R}, x^3 + x - 1 = 0$$

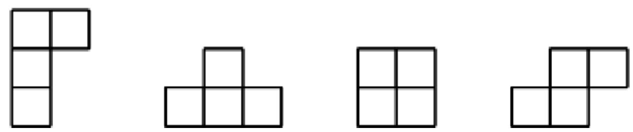
The function $f(x) = x^3 + x - 1$ is a continuous function, and $f(0) = -1$ and $f(1) = 1$. This means there must be an x, $0 < x < 1$, such that $f(x) = 0$. Observe that we could not construct the solution itself, but we were able to prove that it exists.

5. Prove that $x^4 - x - 1 = 0$ has more than one solution.
6. Prove that there exist two irrational numbers x and y such that x^y is rational. *Hint*: Think of the number

$$\left(\sqrt{3}^{\sqrt{2}}\right)^{\sqrt{2}}$$

and consider all possible cases for $\sqrt{3}^{\sqrt{2}}$.

7. Prove by contradiction that the following tiles cannot be put together to make a perfect square. *Hint*: use a parity argument similar to the one we saw in class.



8. Prove the following using the contrapositive:

$$\forall r \in \mathbb{R} - \{1\}, \frac{r}{r-1} \notin \mathbb{Q} \Rightarrow r \notin \mathbb{Q}$$

   Does the statement remain true if we simply reverse the implication?

9. Prove the following is true:

$$\forall n \in \mathbb{N}, n \text{ is even} \Rightarrow \binom{n}{3} \text{ is even}$$

   Hint: If $2x/3$ is an integer, then $x/3$ is an integer because 2 and 3 have no common factors.

10. Which of the following sets is countable and which is uncountable (try your best to explain your answer)?

    - The set of all cups on Earth
    - The set of all real numbers in (0,1)
    - The set of all finite binary sequences
    - The set $\mathbb{R} - \mathbb{Z}$