CSCI 150 Discrete Mathematics
Homework 7

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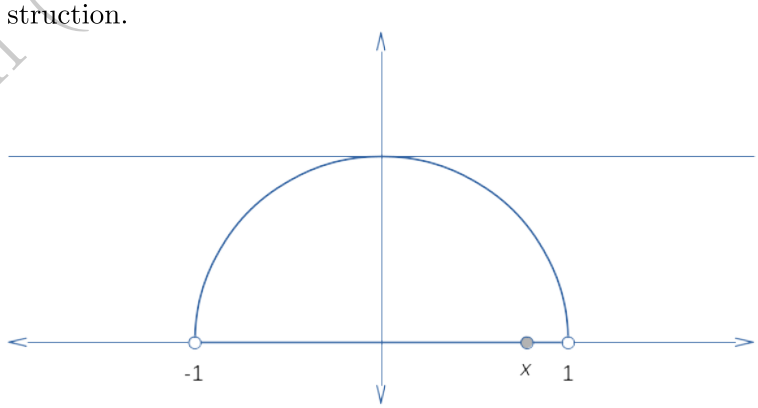
Due Wed. 3/27/2024 (before midnight)

Note: Some ideas in this homework are taken from the book Gentle Introduction to the Art of Mathematics.

1. (Optional, to understand that the interval $(-1, 1)$ is "as big" as the entire set $\mathbb{R}$.)

   - Find a function $f : (-1, 1) \to \mathbb{R}$ that is both one-to-one and onto. This establishes the claim. *Hint*: design your function such that it's continuous and takes -1 to $-\infty$ and $+1$ to ∞.
   - In this approach, we will map $(-1, 1)$ to $\mathbb{R}$ by a geometric construction.



For any $x \in (-1, 1)$:

(a) obtain its vertical projection on the unit circle in the upper half plane, then

(b) make an appropriate projection of that point onto the tangent line (the line $y = 1$), and finally

(c) project vertically onto $\mathbb{R}$ (the line $y = 0$).

Obviously, you only have to figure out (b). Show your work and explain. Extra: can you obtain the function corresponding to this geometric argument?

Extra Challenge: (You are not required to do this) Find a bijection from $[-1, 1]$ to $\mathbb{R}$. This time -1 and 1 are elements of the domain.

2. We learned in class that for any set A, $|A| < |\mathcal{P}(A)|$. For instance, this means that $\mathcal{P}(\mathbb{N})$, the set of all subsets of $\mathbb{N}$ is uncountable. Let $\mathcal{P}_{\mathcal{F}}(\mathbb{N})$ be the set of all **finite** subsets of $\mathbb{N}$.

$$\mathcal{P}_{\mathcal{F}}(\mathbb{N}) = \{X \in \mathcal{P}(\mathbb{N}) \mid X \text{ is finite}\}$$

For instance, the subset $\{2, 4, 6\} \in \mathcal{P}_{\mathcal{F}}(\mathbb{N})$ but the subset $\{1, 3, 5, \ldots\} \notin \mathcal{P}_{\mathcal{F}}(\mathbb{N})$. Show that $\mathcal{P}_{\mathcal{F}}(N)$ is countable.

Note: Here's an ordering that does **not** work: order the elements of $\mathcal{P}_{\mathcal{F}}(\mathbb{N})$ (finite subsets of $\mathbb{N}$) by their size (since each is finite, the size is well defined):

$$\phi, \{1\}, \{2\}, \{3\}, \{4\}, \ldots$$

Do you see the problem?

Hint: Every finite subset has a largest element.

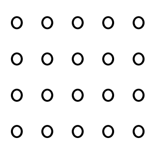
3. We will now show that $\mathcal{P}(\mathbb{N})$ is uncountable even though we already know this fact. This is an opportunity to practice the diagonal method. To do this, we will first represent each subset S of $\mathbb{N}$ by an infinite binary word in which the i^{th} bit is 1 if $i \in S$ and 0 otherwise. To practice this notion try to fill in the table:

infinite binary word	subset of $\mathbb{N}$
$00000\ldots$	
$100000\ldots$	
$011100000\ldots$	
	$\{2,4,6\}$
$1010101010\ldots$	
	$\{3k-2 \mid k \in \mathbb{N}\}$
	$\mathbb{N}$

Now we need to show that there is no bijection from $\mathbb{N}$ to the set of infinite binary words. Reproduce the diagonal argument to show this fact.

Note: Think about this, but you are not required to provide an answer: Why doesn't this diagonal argument disprove the fact that $\mathcal{P}_{\mathcal{F}}(\mathbb{N})$ of question 2 is countable?

4. Consider the following grid of 20 white dots



We color 9 dots black. Prove that three of the black dots make a line.

5. How many numbers in $\{1, 2, \ldots, 546\}$ are not divisible by 2 **and** not divisible by 3 **and** not divisible by 7?

Hint: Negate the requirement, find the answer using inclusion-exclusion, then fix it to answer the original question.

Hint: Check your answer against $546 \times (1/2) \times (2/3) \times (6/7)$ (this does not always work by the way depending on the choice of numbers).

6. Prove by induction that for all integers $n \geq 0$, $\sum_{i=0}^{n}(4i+1) = 2n^2 + 3n + 1$.

Problem 1

A homogeneous subset of $\mathbb{N}$ is one where all the elements have the same parity. A student in CSCI 150 decided to prove that the set all homogeneous subsets of $\mathbb{N}$ is uncountable. The student proposed that every homogeneous subset of $\mathbb{N}$ can be represented as an infinite binary word with all its 1s either in even positions or in odd positions. For instance, $\{1, 3, 5, 7, 9 \ldots\}$ can be represented by the infinite binary word $1010101010\ldots\ldots$, and $\{2, 4, 10\}$ by $0101000001000\ldots$, and $\phi = \{\ \}$ by $000\ldots$, and so on.

With B being the set of all infinite binary words satisfying the above condition, the student mimicked Cantor's diagonalization proof in order to construct a infinite binary word w such that there is no $i \in \mathbb{N}$ with $f(i) = w$. He did this by flipping bits along the diagonal (as shown below).

(Hypothetical function $f : \mathbb{N} \to B$)

$\mathbb{N}$	B
1	$\underline{1}010101010\ldots$
2	$0\underline{1}01000001000\ldots$
3	$00\underline{0}\ldots$
$\vdots$	$\vdots$

$w = 001\ldots$

(a) What's wrong with the student's proof?

(b) Fix the student's proof.

Problem 2

A 3 digit number is good iff it does **not** have 1 in the first digit, **and** does **not** have 2 in the second digit, **and** does **not** have 3 in the third digit. How many good 3 digit numbers are there?

(a) Solve this question using the product rule by identifying the possibilities for each of the three digits.

(b) Do the same using the inclusion-exclusion principle. *Hint*: consider the negation to get the "or" logic. In other words, find the number of bad 3 digit numbers, then adjust your answer to find the number of good 3 digit numbers.

(c) Assume now that all 3 digits must be different. How many 3 digit numbers are good? *Hint*: which technique is more suitable, that of part (a) or part (b)?

Problem 3

There is a $1 \times \ell$ rectangle. For simplicity, we will assume that ℓ is a positive integer.

(a) We place $\ell+1$ points in the rectangle. Prove that two of the points must be within a distance of $\sqrt{2}$.

(b) Show by an example that you can place $\ell+1$ points in the rectangle such that the distance between any two of them is at least $\sqrt{2}$.

(c) Show by an example that you can place ℓ points in the rectangle such that no two of them are within a distance of $\sqrt{2}$.