Problem 1: A recurrence
Consider the following recurrence:

\[ a_n = a_{n-1} - a_{n-2} \]

where \( a_0 = 0 \) and \( a_1 = 1 \).

(a) Using the recurrence and the the initial conditions, generate the first 18 numbers of the sequence \( \{a_n\} \). Try to guess a way to compute \( a_n \) immediately by simply knowing \( n \).

(b) Solve for \( a_n \). Hint: observe that \( a_n \) has the form \( a_n = Aa_{n-1} + Ba_{n-2} \), but you are going to encounter a little surprise!

(c) Your expression for \( a_n \) in part (b) most likely contains the imaginary number \( i \). Use the binomial theorem to obtained a nicer expression for \( a_n \):

\[ a_n = \frac{1}{2^{n-1}} \left[ \binom{n}{1} 3^0 - \binom{n}{3} 3^4 + \binom{n}{5} 3^2 - \ldots \right] \]

Problem 2: An interesting language
Consider a language that uses the alphabet \{0, 1, \#\}. In this language words obey one single rule: a \# cannot follow a \#. How many words of length \( n \) exist in this language? Hint: choose the first letter to establish a recurrence.

Problem 3: Stirling numbers
Consider the problem of partitioning a set of size \( n \) into \( k \) sets. Let \( S(n,k) \) be the number of possible partitions. For instance \( S(3,2) = 3 \):

\[ \{ \{1\}, \{2,3\} \} \quad \{ \{2\}, \{1,3\} \} \quad \{ \{3\}, \{1,2\} \} \]

(a) Argue that the number of partitions in which the first element forms a set by itself is \( S(n-1, k-1) \).

(b) Argue that the number of partitions in which the first element does not form a set by itself is \( kS(n-1,k) \).
(c) Conclude that:

\[ S(n, k) = S(n-1, k-1) + kS(n-1, k) \]

and fill the following table showing \( S(n, k) \) for different values of \( n \) and \( k \).

<table>
<thead>
<tr>
<th>( n \backslash k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(d) In how many ways can we place 7 labeled balls into 4 bins if:
- bins are indistinguishable and bins may be empty
- bins are indistinguishable and no bin can be empty
- bins are distinguishable and no bin can be empty

*Hint:* use the idea of \( S(n, k) \) and you might want to consult the last section in Note 6.

**Problem 4: Catalan numbers**

Catalan numbers satisfy the following:

- \( c_0 = 1 \)
- \( c_n = c_0c_{n-1} + c_1c_{n-2} + \ldots + c_{n-1}c_0 \)
- \( c_n = \frac{1}{n+1} \binom{2n}{n} \)

Search the internet for Catalan numbers and find few examples that satisfy that recurrence. Pick one and describe it.