CSCI 150 Discrete Mathematics
Homework 8

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Due Wed. 4/3/2024 (before midnight)

1. (This is somewhat piegonhole in reverse) A person wants to collect c identical coupons. There are n different types of coupons. How large must m be so that if this person collects m coupons, he/she is guaranteed to obtain c identical ones? (m must be a function of c and n).

2. My drawer has n pairs of socks (so $2n$ socks). The socks came back from the laundry, surprisingly with no sock missing! But they are all separated. Each sock is labeled either "left" or "right". Assume that I grab some socks blindly, how many do I need to guarantee a good pair?

   - If I must use a correct pair of socks (two that match). Show how to use the concept of pigeonhole to find the answer.
   - If I am happy to use any left sock with any right sock (this is not pigeonhole, but it has some logic to it).

3. Consider a binary word of length 70. Prove that there are at least two occurrences of some sequence of 6 bits.

4. There is a contest with 40 Pokemons. There are 18 Pokemons who like to fight in the sky, and 23 who like to fight on ground. Several of them like to fight in the water. The number of those who like to

fight in the sky and on ground in 9. There are 7 Pokemons who like to fight in the sky and in water, and 12 who like to fight on ground and in water. There are 4 Pokemons who like to fight in the sky, on ground, and in water. How many Pokemons like to fight in water?

5. In a class of 20, 12 are boys and 13 wear glasses. There are twice as many boys with glasses as girls with no glasses. How many girls wear glasses?

6. Consider the following well-known Fibonacci sequence given by

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

where $F_0 = 0$ and $F_1 = 1$ and

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2$$

Prove the following two statements by induction:

- For all $n \geq 0$, $\sum_{i=0}^{n} F_i = F_{n+2} - 1$
- For all $n \geq 1$, $F_n \geq \phi^{n-2}$ where $\phi = (1 + \sqrt{5})/2$. *Hint*: ϕ is the solution of a well-know quadratic equation.

One of the above requires strong induction and multiple base cases (which one and why?).

Problem
Assume that $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

(a) Suppose that we change the induction mechanism as follows:

- Base case: Prove that $P(0)$ is true
- Inductive step: Prove that for all $k \geq 0, P(k) \Rightarrow P(k + 2)$

Explain why this would not constitute a valid proof that $P(n)$ is true for all $n \in \mathbb{N}$. How would you change the base case to obtain a valid proof?

(b) Use the above strategy to prove the following statement:

$$\forall n \in \mathbb{N}, n^2 = 4m \vee n^2 = 4m + 1$$

where $k \in \mathbb{N}$.

(c) Prove the same statement using a typical strong induction mechanism (it should not be very different from above).