CSCI 150 Discrete Mathematics
Homework 9

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Due Thu. 4/18/2024 (before midnight)

1. An alien species communicates using a three-letter alphabet $\{x, y, z\}$. In their language, words must obey one rule: zz cannot be part of any word; otherwise, the speaker will go to sleep and never finish the sentence. Describe the number of words of length n by a recurrence. Let a_n be the number of words of length n, and express a_n in terms of a_{n-1} and a_{n-2}.

   Hint: Do as we did with the tiling problem, i.e. consider different cases based on how you start a word, then for each case figure out in how many ways you can finish it.

2. Consider a version of the Tower of Hanoi where each disk is duplicated, so we have $2n$ disks with 2 disks of each size. The rules of the game are the same. Let a_n be the number of moves needed to solve this $2n-$disk problem.

   (a) Find a recurrence for a_n.

   (b) Guess a solution for a_n is terms of n (by exploring), and prove it by induction.

3. Consider a sequence where $a_0 = 1$, $a_1 = -2$, and $a_n = -2a_{n-1} - a_{n-2}$ for $n \geq 2$.

   (a) Guess a_n as a function of n and prove it by strong induction.

   (b) Use the characteristic equation method.

4. Consider the following recurrence,

$$a_n = \frac{1}{2}a_{n-1} + 1$$

   where $a_1 = 1$.

   (a) Guess a pattern for a_n and prove it by induction.

   (b) Convert the recurrence for a_n into the form $a_n = Aa_{n-1} + Ba_{n-2}$ by eliminating the constant 1 in the recurrence. Solve for a_n using the characteristic equation.

   (c) Find a_n using the generating function method (follow the example illustrated in class).

5. Consider $2n$ points on the circumference of a circle. In how many ways can we join the points pairwise by n chords such that no two chords intersect? Call this number a_n, find a recurrence for it, then solve it.

Problem 1 (optional)
Consider the following sequences starting at $a_0, a_1, \ldots$:

$$5, -10, 20, -40, \ldots$$

$$1, 7, 49, 343, \ldots$$

(a) For each of the sequences above, find a recurrence of the form $a_n = Aa_{n-1}$ for $n \geq 1$, and solve for a_n as a function of n.

(b) For each of the sequences above, find a recurrence of the form $a_n = Aa_{n-1} + Ba_{n-2}$ for $n \geq 2$, by considering the recurrence from part (a) for a_n and a_{n-1}; the solution is not unique, depending on how you combine recurrences, so find the solution that corresponds to adding up the recurrences.

(c) There are infinitely many recurrences of the form $a_n = Aa_{n-1} + Ba_{n-2}$ that work since we can write $a_n = cp^n + 0 \cdot q^n$ for $q \neq p$. Find a recurrence of the form $a_n = Aa_{n-1} + Ba_{n-2}$ for $n \geq 2$ that works for both sequences at the same time.

Problem 2 (optional)
Consider the following recurrence:

$$a_n = a_{n-1} - a_{n-2}$$

where $a_0 = 0$ and $a_1 = 1$.

(a) Using the recurrence and the initial conditions, generate the first 18 numbers of the sequence $\{a_n\}$. Try to guess a way to compute a_n immediately by simply knowing n.

(b) Solve for a_n. *Hint*: observe that a_n has the form $a_n = Aa_{n-1} + Ba_{n-2}$.

(c) Your expression for a_n in part (b) will contain the imaginary number i. Use the binomial theorem to obtain a nicer expression for a_n:

$$a_n = \frac{1}{2^{n-1}}\left[\binom{n}{1}3^0 - \binom{n}{3}3^1 + \binom{n}{5}3^2 - \ldots\right]$$