Exercise 1
Consider the well-known Fibonacci sequence given by

\[0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots\]

where \(F_0 = 0\) and \(F_1 = 1\) and

\[F_n = F_{n-1} + F_{n-2}, n \geq 2\]

We want to show the property that for all \(n \geq 0\):

\[\sum_{i=0}^{n} F_i = F_{n+2} - 1\]

Prove this by induction. When you do your inductive step, figure out what value for \(n_0\) is appropriate. Recall \(n_0\) must satisfy:

\[\forall k \geq n_0, P(k) \Rightarrow P(k+1)\]

Note: Does the property also hold for \(n = -1\)? What do you think?

Exercise 2
Consider the recurrence

\[a_n = 2a_{n-1} + 2a_{n-2}, n \geq 2\]

where \(a_0 = 1\) and \(a_1 = 3\).
Prove (by strong induction) that
\[ a_n = \left( \frac{1}{2} + \frac{\sqrt{3}}{3} \right) \rho^n + \left( \frac{1}{2} - \frac{\sqrt{3}}{3} \right)(2 - \rho)^n \]
where \( \rho = 1 + \sqrt{3} \).

*Hint:* Knowing that \( \frac{2}{\rho} + \frac{2}{\rho^2} = \frac{2}{2 - \rho} + \frac{2}{(2 - \rho)^2} = 1 \) will simplify your inductive step [this is very similar to the Fibonacci problem we did in class].

*Note:* Explain why the base case must cover \( n = 0 \) and \( n = 1 \) (so \( a_0 = 1 \)).

**Exercise 3**
Prove (by strong induction) the following:
\[ \forall n \geq 12, \ n = 4x + 5y, \text{ where } x \text{ and } y \text{ are non-negative integers} \]

*Note:* Explain why you need so many base cases. How many?

**Exercise 4**
An alien species communicates using a three-letter alphabet \( \{x, y, z\} \). In their language, words must obey one rule: \( zz \) cannot be part of any word; otherwise, the speaker will go to sleep and never finish the sentence. Describe the number of words of length \( n \) by a recurrence. Let \( a_n \) be the number of words of length \( n \), and express \( a_n \) in terms of \( a_{n-1} \) and \( a_{n-2} \).

*Hint:* Do as we did with the tiling problem, i.e. consider different cases based on how you start a word, then for each case figure out in how many ways you can finish it.

**Exercise 5**
Consider a version of the Tower of Hanoi where each disk is duplicated, so we have \( 2n \) disks with 2 disks of each size. The rules of the game are the same. Let \( a_n \) be the number of moves needed to solve this \( 2n \)-disk problem.

(a) Find a recurrence for \( a_n \).

(b) Guess a solution for \( a_n \) in terms of \( n \) (by exploring), and prove it by induction.

**Exercise 6**
Consider a sequence where \( a_0 = 1 \), \( a_1 = -2 \), and \( a_n = -2a_{n-1} - a_{n-2} \) for \( n \geq 2 \). Guess \( a_n \) as a function of \( n \) and prove it by strong induction.
Problem 1

Consider the following recurrence

\[ a_n = \frac{1}{2}a_{n-1} + 1 \]

where \( a_1 = 1 \).

(a) Guess a pattern for \( a_n \) and prove it by induction.

(b) Convert the recurrence for \( a_n \) into the form \( a_n = Aa_{n-1} + Ba_{n-2} \) by eliminating the constant 1 in the recurrence. Solve for \( a_n \) using the characteristic equation.

(c) Find \( a_n \) using the generating function method (follow the example illustrated in class).

The next two problems are optional for those you want to explore further...

Problem 2

Consider the following recurrence:

\[ a_n = a_{n-1} - a_{n-2} \]

where \( a_0 = 0 \) and \( a_1 = 1 \).

(a) Using the recurrence and the initial conditions, generate the first 18 numbers of the sequence \( \{a_n\} \). Try to guess a way to compute \( a_n \) immediately by simply knowing \( n \).

(b) Solve for \( a_n \). \textit{Hint:} observe that \( a_n \) has the form \( a_n = Aa_{n-1} + Ba_{n-2} \).

(c) Your expression for \( a_n \) in part (b) will contain the imaginary number \( i \). Use the binomial theorem to obtained a nicer expression for \( a_n \):

\[ a_n = \frac{1}{2^{n-1}} \left[ \binom{n}{1} 3^0 - \binom{n}{3} 3^1 + \binom{n}{5} 3^2 - \ldots \right] \]
Problem 3
Consider the following sequences starting at $a_0, a_1, \ldots$:

$5, -10, 20, -40, \ldots$

$1, 7, 49, 343, \ldots$

(a) For each of the sequences above, find a recurrence of the form $a_n = Aa_{n-1}$ for $n \geq 1$, and solve for $a_n$ as a function of $n$.

(b) For each of the sequences above, find a recurrence of the form $a_n = Aa_{n-1} + Ba_{n-2}$ for $n \geq 2$, by considering the recurrence from part (a) for $a_n$ and $a_{n-1}$; the solution is not unique, depending on how you combine recurrences, so find the solution that corresponds to adding up the recurrences.

(c) There are infinitely many recurrences of the form $a_n = Aa_{n-1} + Ba_{n-2}$ that work since we can write $a_n = cp^n + 0 \cdot q^n$ for $q \neq p$. Find a recurrence of the form $a_n = Aa_{n-1} + Ba_{n-2}$ for $n \geq 2$ that works for both sequences at the same time.