Problem 1
Set union and intersection are associative, commutative, and distributive. Shown below with the logical counterpart, where $A$, $B$, and $C$ are sets; $P$, $Q$, and $R$ are propositions. For instance, you can think of $P$ as $x \in A$. Make sure you understand these.

**Associative**

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(P \lor Q) \lor R = P \lor (Q \lor R)$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

$$(P \land Q) \land R = P \land (Q \land R)$$

**Commutative**

$$A \cup B = B \cup A$$

$$P \lor Q = Q \lor P$$

$$A \cap B = B \cap A$$

$$P \land Q = Q \land P$$

**Distributive**

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$P \land (Q \lor R) = (P \land Q) \lor (P \land R)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$P \lor (Q \land R) = (P \lor Q) \land (P \lor R)$$
Problem 2
Show that

\[ A^C \cup B^C = (A \cap B)^C \]
\[ A - (B \cap C) = (A - B) \cup (A - C) \]

where

\[ A^C = \{x : x \notin A\} \]
\[ A - B = \{x : x \in A \land x \notin B\} \]

Solution:
Show \( A^C \cup B^C \subset (A \cap B)^C \):

\[ x \in (A^C \cup B^C) \Rightarrow (x \in A^C) \lor (x \in B^C) \Rightarrow (x \notin A) \lor (x \notin B) \]
\[ \Rightarrow (\neg(x \in A)) \lor (\neg(x \in B)) \Rightarrow (\text{De Morgan}) \Rightarrow (x \notin (A \cap B)) \Rightarrow x \notin (A \cap B) \Rightarrow x \in (A \cap B)^C \]

Show \((A \cap B)^C \subset A^C \cup B^C\):

Reversing the above implications works.

Show \( A - (B \cap C) \subset (A - B) \cup (A - C) \):

\[ x \in A - (B \cap C) \Rightarrow x \in A \land x \notin (B \cap C) \Rightarrow x \in A \land \neg(x \in (B \cap C)) \]
\[ \Rightarrow x \in A \land (\neg(x \in B) \lor \neg(x \in C)) \Rightarrow (\text{De Morgan}) \Rightarrow x \in A \land (\neg(x \in B) \lor \neg(x \in C)) \]
\[ \Rightarrow (\text{Distributive}) \Rightarrow (x \in A \land \neg(x \in B)) \lor (x \in A \land \neg(x \in C)) \]
\[ \Rightarrow (x \in A - B) \lor (x \in A - C) \Rightarrow x \in (A - B) \cup (A - C) \]

Show \((A - B) \cup (A - C) \subset A - (B \cap C)\):

Reversing the implications above works.

Problem 3
Which of the following sets is countable and which is uncountable?

- The set of all irrational numbers in \([0, 1]\)
- The set of all sets of numbers that are divisible by 17
- The set of all books, where “book” is a finite sequence of symbols in \{a,...,z,A,...,Z,0,...,9,.,:,?,!\”\}
- The set of all finite sets of book
Solution: The set of irrationals in $[0,1]$ is uncountable. To see this, we first argue that the set of real numbers in $[0,1]$ is uncountable using the exact same diagonal method illustrated in class. Then consider the set of all rational numbers in $[0,1]$, this set is countable because it’s a subset of $\mathbb{Q}$. If our set were countable, then it’s union with $\mathbb{Q}$ would be countable, and thus the set of all reals in $[0,1]$ would be countable, a contradiction.

The second set is uncountable. First observe that it’s the power set of $$\{0, 17, -17, 34, -34, \ldots\}$$ which is countable (easy to show by finding an order with a finite rank for each element). Since the power set is larger the its set, and both are infinite, it cannot be countable (otherwise, they would have the same size).

The third set is countable. Simply order the books by their length, and within each category, order them alphabetically. This gives an order on the books in which each book has a finite rank.

The fourth set is also countable. Given the order of books above, observe that there is a finite number of finite sets that contain book $i$ as the largest book, namely $2^{i-1}$ such sets (where each book below $i$ is either in the set or not). This means ordering the finite sets of books by their largest book gives a finite rank for each set.

**Problem 4**

- Assume every person has less than 500,000 strands of hair on his/her head. In a city with more than 9.5 million people, show that we can find 20 with the exact same number of strands of hair on their heads.

**Solution:** The possible numbers of strands are in $\{0, \ldots, 499000\}$. By pigeonhole, we must have at least $$\left\lceil \frac{9500001}{500000}\right\rceil = 20$$ people with the same number of strands.

- We select 38 positive integers, all less than 1000. Show that there will be two of them whose difference is at most 26.

**Solution:** Consider the following “boxes” by dividing $\{1, \ldots, 999\}$ into 37 parts (by design):

$$\{1, \ldots, 27\}$$
$$\{28, \ldots, 54\}$$
$$\vdots$$
$$\{973, \ldots, 999\}$$
We have 37 boxes. Each of the 38 numbers must belong to some box. By pigeonhole, at least
\[ \left\lceil \frac{38}{37} \right\rceil = 2 \]
numbers must belong to the same box. But the smallest and largest numbers in each box have a difference of 26. This proves the claim.

• Show by a counter example that, in general, the following is not true:

\[ \left\lfloor \frac{m-1}{n} \right\rfloor + 1 = \left\lceil \frac{m}{n} \right\rceil \]

Solution: This is a counter example: \( m = 2.5, n = 2 \). Here’s the thought process that produced it: Let’s say \( m/n \) falls between two integers \( i \) and \( i+1 \), so \( m/n = i + \epsilon \), and \( \lceil m/n \rceil = i + 1 \). Now \( (m-1)/n = m/n - 1/n \). If subtracting \( 1/n \) from \( m/n \) puts us between \( i-1 \) and \( i \), then \( \lfloor (m-1)/n \rfloor + 1 = i \) and we are done. This can be done if \( 1/n > \epsilon \). Let \( n = 2 \) (so \( 1/n = 0.5 \)) and \( \epsilon = 0.25 \). Now \( m/n = i + 0.25 \). Any value of \( i \) will work, so let \( m/n = 1.25 \), then \( m = 2.5 \).