Saad Mneimneh, Computer Science, Hunter College of CUNY

These are sample Test 2 questions that appeared in previous years

Problem 1: Proof techniques I

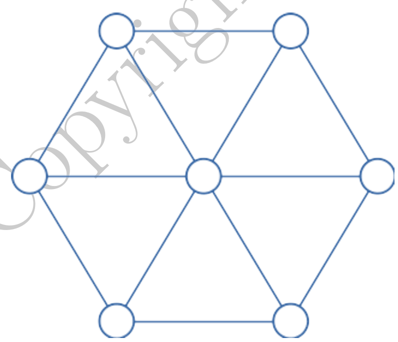
(a) (**3 points**) Let x and y be integers. Prove the following statement using the *contrapositive*:

$$x + y \text{ is odd} \Rightarrow x \neq y$$

State the contrapositive first (**1 point**), then prove it.

(b) (**3 points**) Prove by *contradiction* the following statement:

One cannot place the numbers 1, 2, 3, 4, 5, 6, and 7
to make the sum of the numbers on the hexagon
equal to 7 times the number in the center.



Let a, b, c, d, e, f be the numbers on the hexagon, and g the number in the center. Write down an equation that corresponds to the negation of the statement (**1 point**), then try to reach a contradiction.

Note: This statement can be easily proved by explicitly trying all possible values for g. Such proof is not going to get credit.

Problem 2: Proof techniques II

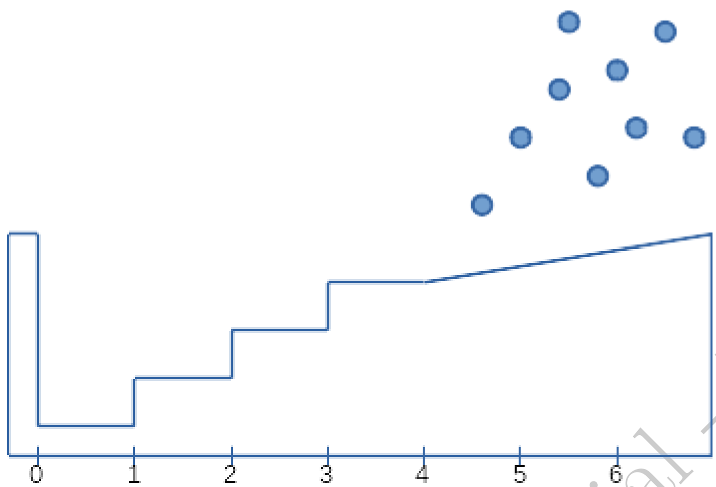
(a) (3 points) Prove using induction that $P(n)$ below is true for every $n \in \mathbb{N} = \{0, 1, 2, 3 \ldots\}$:

$$P(n) : \prod_{i=1}^{n} \left(1 + \frac{2}{i}\right) = \frac{(n+1)(n+2)}{2}$$

Give the base case and express $P(k)$ and $P(k+1)$ (**1 point**), then show the work for the proof by induction.

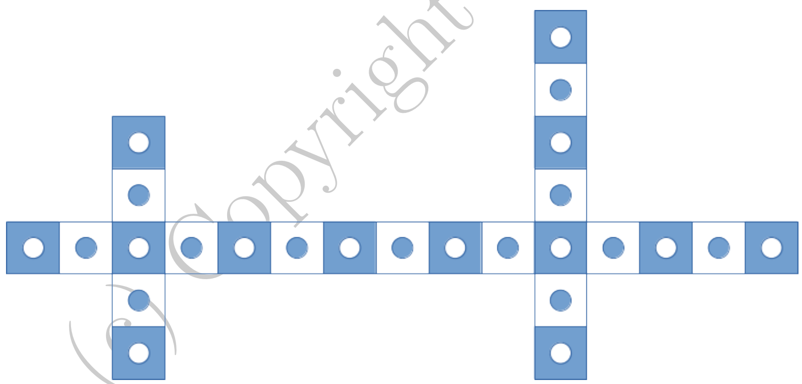
Problem 3: Proof techniques III

(a) **(3 points)** The balls fall as shown. The balls settle when each one is touching a horizontal surface. Show that when the balls settle, three of them must be within a distance of at most 1 from each other (you get **1 point** for choosing the correct proof technique).



(b) (**3 points**) In the following game, there are 25 dots (and 25 squares). The rule of the game is that every dot must move to an adjacent square. Prove that no matter how the dots are moved, a square will contain more than one dot.

You get (**1 point**) for choosing the correct proof technique.



Problem 4: Odd powers and induction

Suppose we want to prove that a property P is true for every integer in $\mathbb{N}_{odd} = \{1, 3, 5, 7, 9, \ldots\}$. Consider the following induction mechanism:

1. Base case: Verify the property $P(1)$
2. Inductive step: Prove that for all $k \geq 1$, $P(k) \Rightarrow P(k + 1)$

(a) (2 points) Why might the above mechanism not constitute a valid proof?

(b) (2 points) How would you modify the inductive step to obtain a valid proof?

(c) (4 points) Use your modified mechanism to prove that every integer $n \in \mathbb{N}_{odd}$ satisfies $2^n + 3^n = 5m$, where m is an integer.

Problem 5: One hundred and one dalmatians

Roger has 101 dalmatian dogs. Each dog has a unique number of black spots from the set $\{1, 2, 3, \ldots, 101\}$. We choose any 52 of the 101 dogs. We want to prove that any set of 52 dogs satisfies the following addition property:

Addition property: The numbers of spots on two of the dogs add up to exactly the number of spots on some other dog.

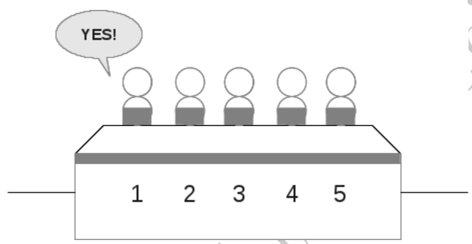
(a) (2 points) Which proof technique is most appropriate and why?

(b) (4 poins) Prove that any set of 52 dogs has the addition property. *Hint*: First, prove the statement assuming that one of the 52 chosen dogs has 101 spots (this will get you 3 out of the 4 points). Then generalize.

(c) (4 points) How many among the 101 dogs have an odd number of spots that is not divisible by 3 and not divisible by 5. Show your work.

Problem 6: Judges

Five judges sitting in a row must each give a **Yes/No** vote. Each will announce his/her vote in turn, starting from the first judge.



(**4 points**) Find the number of ways the judges can vote without having any consecutive **Yes, No, Yes**. You must use the principle of Inclusion-Exclusion and convert this into an OR logic. In other words, first find the number of ways in which some three judges i, $i+1$, and $i+2$ vote **Yes**, **No**, **Yes**.

To do so, define A_i to be the set of votes for which judge i and judge $i+1$ and judge $i+2$ vote **Yes**, **No**, **Yes**. Then find $|A_1 \cup A_2 \cup A_3|$ and subtract from the total number of possible votes. Write your answer is the following format:

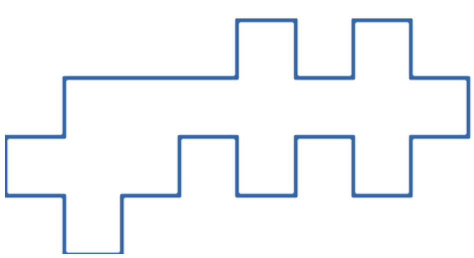
$$\square - (\square + \square + \square - \square - \square - \square + \square)$$

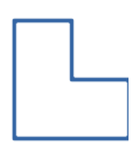
Note 1: As a reminder, you must find: $|A_1|$, $|A_2|$, $|A_3|$, $|A_1 \cap A_2|$, $|A_1 \cap A_3|$, $|A_2 \cap A_3|$, and $|A_1 \cap A_2 \cap A_3|$.

Note 2: Partial credit will be given for other methods not involving Inclusion-Exclusion.

Problem 7: Tiling

(2 points) Prove by contradiction that the shape on the left cannot be covered by shapes on the right (rotations allowed). *Hint*: Chessboard coloring argument.





Problem 8: Homegeneous sets

A homogeneous subset of $\mathbb{N}$ is one in which all integers have the same parity. Every homogeneous subset of $\mathbb{N}$ can be represented as an infinite binary word with all its 1s either in even positions or in odd positions. For instance, $\{1, 3, 5, 7, 9 \ldots\}$ can be represented by the infinite binary word $1010101010\ldots\ldots$, and $\{2, 4, 10\}$ by $0101000001000\ldots$, and $\phi = \{\ \}$ by $000\ldots$, and so on.

With B being the set of all infinite binary words satisfying the above condition, here's a Cantor's diagonalization proof in order to construct a infinite binary word w such that there is no $i \in \mathbb{N}$ with $f(i) = w$. The word w is constructed by flipping bits along the diagonal (as shown below).

(Hypothetical function $f : \mathbb{N} \to B$)

$\mathbb{N}$	B
1	$\underline{1}010101010\ldots$
2	$0\underline{1}01000001000\ldots$
3	$00\underline{0}\ldots$
$\vdots$	$\vdots$

$w = 001\ldots$

(a) (2 points) What's wrong with the proof?

(b) (2 points) Fix the proof or provide your own proof that the set of all homogeneous subsets of $\mathbb{N}$ is uncountable.