Answers to Sample Questions

Problem 1

(a) $x+y$ is odd $\Rightarrow x \neq y$

contrapositive: $x = y \Rightarrow x+y$ is even

proof: $x=y \Rightarrow x+y = x+x = 2x \Rightarrow x+y$ is even.

(b) Let the numbers be a, b, c, d, e, f, and g.

To prove by contradiction, assume

$$7g = a+b+c+d+e+f$$

then $8g = a+b+c+d+e+f+g = 1+2+\cdots+7 = 28$

then $g = \frac{28}{8} \notin \mathbb{N}$, a contradiction.

Problem 2.

Base case. $n=0$. $\prod_{i=1}^{0}\left(1+\frac{2}{i}\right) = 1$ (empty product)

but $\frac{(0+1)(0+2)}{2} = 1$ as well. ✓

Inductive hypothesis. $\prod_{i=1}^{k}\left(1+\frac{2}{i}\right) = \frac{(k+1)(k+2)}{2}$

Inductive hypothesis. $\forall\, k \geq 0,\ P(k) \Rightarrow P(k+1)$

$P(k+1)$: $\prod_{i=1}^{k+1}\left(1+\frac{2}{i}\right) = \frac{(k+2)(k+3)}{2}$

$$\prod_{i=1}^{k+1}\left(1+\frac{2}{i}\right) = \left[\prod_{i=1}^{k}\left(1+\frac{2}{i}\right)\right]\cdot\left(1+\frac{2}{k+1}\right) = \frac{(k+1)(k+2)}{2}\left(1+\frac{2}{k+1}\right)$$

$$= \frac{(k+1)(k+2)}{2}\cdot\frac{k+3}{k+1} = \frac{(k+2)(k+3)}{2}.$$

Problem 3.

(a) There are 4 horizontal surfaces and 9 balls. When the 9 balls settle on horizontal surfaces, one of them must have at least $\lceil \frac{9}{4} \rceil = 3$ balls. Therefore, some 3 balls must be within a distance of 1 foot from each other.

(b) There are 13 white balls and 12 white squares. After the move, all white balls will be in white squares. By pigeonhole, one square must contain at least 2.

Problem 4

(a) The mechanism is not valid because P(2) may not be true, so it's not guaranteed that one can prove $P(1) \Rightarrow P(2)$. Even though P(2) itself is not relevant, but P(3) can only be proved using $P(2) \Rightarrow P(3)$. If P(2) is not true, one cannot prove P(3) true.

(b) We can modify the mechanism to use $P(k) \Rightarrow P(k+2)$ instead.

(c)

Given $P(k)$ is true, i.e. $2^k + 3^k = 5m$

let's prove $P(k+2)$ is true.

$$P(k+2): \quad 2^{k+2} + 3^{k+2} = 5m'$$

$$\begin{aligned} 2^{k+2} + 3^{k+2} &= 2^2.2^k + 3^2.3^k = 4.2^k + 9.3^k \\ &= 4.2^k + 4.3^k + 5.3^k \\ &= 4(2^k + 3^k) + 5.3^k \\ &= 4.5m + 5.3^k = 5\underbrace{(4m + 3^k)}_{m'}. \end{aligned}$$

<u>Problem 5.</u>

(a),(b) If dog 101 is among the choices, then we have 51 dogs in $\{1, 2, \ldots, 100\}$. Consider the following "boxes":

$$\{1,100\}, \{2,99\}, \{3,98\}, \ldots, \{50,51\}$$

These are 50 boxes. By <u>pigeonhole</u>, two dogs must belong to the same box, and by construction, this means they add up to 101.

If dog 101 is not among the choices, let x be the largest # dots. Consider the following "boxes":

$$\{1, x-1\} \quad \{2, x-2\}, \ldots \qquad \text{As before,}$$

we have $\lceil \frac{x-1}{2} \rceil$ boxes and 51 dogs. By pigeonhole one box must have at least $\left\lceil \frac{51}{\lceil \frac{x-1}{2} \rceil} \right\rceil$ dogs. The worst case is when $x = 101$, which gives 2.

(c) We want the number of integers in $\{1, \ldots, 101\}$ that are not divisible by 2 and not divisible by 3 and not divisible by 5. If we count the opposite, we can use inclusion-Exclusion:

$$|S_2| = \left\lfloor \frac{101}{2} \right\rfloor = 50 \qquad |S_2 \cap S_5| = \left\lfloor \frac{101}{10} \right\rfloor = 10$$

$$|S_3| = \left\lfloor \frac{101}{3} \right\rfloor = 33 \qquad |S_3 \cap S_5| = \left\lfloor \frac{101}{15} \right\rfloor = 6$$

$$|S_5| = \left\lfloor \frac{101}{5} \right\rfloor = 20 \qquad |S_2 \cap S_3 \cap S_5| = \left\lfloor \frac{101}{30} \right\rfloor = 3$$

$$|S_2 \cap S_3| = \left\lfloor \frac{101}{6} \right\rfloor = 16$$

So $|S_2 \cup S_3 \cup S_5| = 50+33+20-16-10-6+3 = 74$

Therefore, the answer is $101 - 74 = 27$.

Problem 6.

YES NO YES * * : $|A_1| = 4$

* YES NO YES * : $|A_2| = 4$

* * YES NO YES: $|A_3| = 4$

$|A_1 \cap A_2| = 0$ (can't have judge 2 say YES and NO)

$|A_2 \cap A_3| = 0$ (same)

$|A_1 \cap A_3| = 1$: YES NO YES NO YES

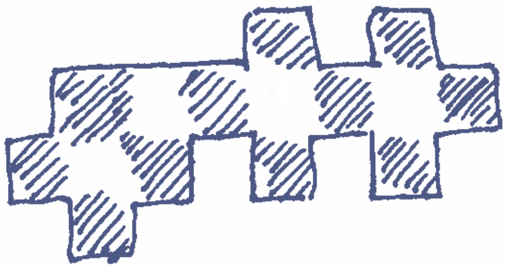
$|A_1 \cap A_2 \cap A_3| = 0.$

So we have: $\underbrace{4+4+4-0-0-1+0}_{|A_1 \cup A_2 \cup A_3|} = 11$ (inclusion-Exclusion)

The answer is given by subtracting from total: $2^5 - 11 = 21.$

Problem 7:

Let's use a chess board coloring.



There are 4 white square.

Assume that we can cover this shape by Trominos. Then we need exactly 5 Trominos. Each tromino will have at least one white square.

Therefore, after covering we have at least 5 white squares, a contradiction.

Problem 8

(a) w constructed in this way may not be in B.

(b) we can fix it by changing pairs of bits.

$00 \to 01$
$01 \to 00$
$10 \to 00$

This way, all 1s are in even positions and it's guaranteed that $w \in B$. This completes the diagonalization argument and proves that B is uncountable.