Summary of Proofs, so far...

By contradiction:

$$P \Rightarrow \dots \Rightarrow \text{False}$$

establishes that P is False, so start with the negation of the statement you want to prove

The contrapositive:

If you want to prove a statement of the form

$$P \Rightarrow Q$$

and you get stuck going from "left to right", then consider the contrapositive

$$\neg Q \Rightarrow \neg P$$

Note: Both techniques require the skill of being able to negate a statement.

Direct proof:

If you want to prove P is true, start with some R that you know is true, and prove

$$R \Longrightarrow P$$

Warning: If you want to prove P is true, and you start with P and reach something you know is true, that's **not** a proof

$$\underbrace{P \Rightarrow \cdots \Rightarrow \text{True}}_{\text{does not establish } P \text{ is true}}$$

unless all implications can be reversed.

Example: Prove $1=2$

$$\underbrace{1=2}_{P} \Rightarrow 1=2 \wedge 2=1 \Rightarrow \underbrace{3=3}_{\text{True}}$$

(by adding on each side of equality)

reversible

not reversible

Example: Prove $\forall a,b \geqslant 0,\ \frac{a+b}{2} \geqslant \sqrt{ab}$

is this reversible? Yes if $a,b \geqslant 0$

$$\frac{a+b}{2} \geqslant \sqrt{ab} \Leftrightarrow a+b \geqslant 2\sqrt{ab} \Rightarrow (a+b)^2 \geqslant 4ab$$

$$\Leftrightarrow a^2+b^2+2ab \geqslant 4ab \Leftrightarrow a^2+b^2-2ab \geqslant 0$$

$$\Leftrightarrow \underbrace{(a-b)^2 \geqslant 0}_{\text{True}}$$

Here, all implications are reversible