Problem 4.12

Show that

$$\{ \langle G \rangle \mid G \text{ is a CFG over } \{0,1\}^* \text{ and } 1^* \subseteq L(G) \}$$

is decidable.

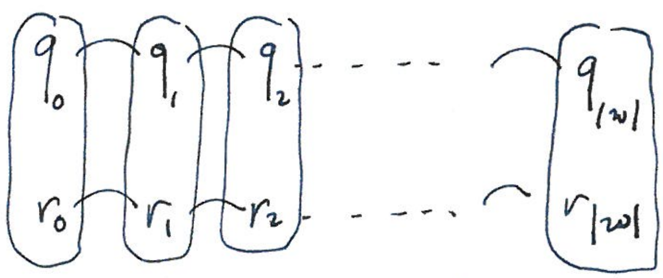
The language 1^* is regular. The intersection of a CFL and a regular language is CFL. We can construct a CFG for that intersection and check if it's empty or not.

Problem 4.16

Consider a string w such that

$$w \in L(M_1)$$
$$w \notin L(M_2)$$

If $|w| \geq |Q_{M_1}| |Q_{M_2}|$, then consider the sequence of states in M_1 and M_2 on input w



Since we have at least $|Q_{M_1}| \times |Q_{M_2}|$ transitions, we have at least $1 + |Q_{M_1}||Q_{M_2}|$ pairs of states in M_1 and M_2 as shown above.

Therefore, some pair repeats and w can be truncated to a smaller string that ends in the same last states in M_1 and M_2. So we have a smaller string on which M_1 and M_2 disagree. Therefore, it's enough to test all w such that $|w| < |Q_{M_1}| \times |Q_{M_2}|$. If M_1 and M_2 agree on all these, they can't disagree on longer ones.

Problem 4.17

Prove that C is Turing-recognizable iff a decidable language D exists such that

$C = \{x \mid \exists y (\langle x,y \rangle \in D)\}$

If C is Turing recognizable then there is a TM M that accepts it. Let $D = \{\langle x,y \rangle \mid M \text{ accepts } x \text{ in } y \text{ steps}\}$ clearly D is decidable, and if $x \in C$, then $\exists y$ such that $\langle x,y \rangle \in D$.

If there exists a decidable D such that $C = \{x \mid \exists y (\langle x,y \rangle \in D)\}$ Then build a TM that goes over all y in order and simulate D's machine on $\langle x, y \rangle$. If $\langle x,y \rangle$ is accepted for some y, accept. So C is Turing-recognizable.