1. Prove, using the Pumping Lemma, that $\{a^n ba^{2n} \mid n > 0\}$ is not regular.

   Let N be the constant of the lemma. Let w be the word $a^N ba^{2N}$. By the lemma, there exist x, y, and z such that w = xyz, |xy| <= N, |y| > 0, and for all k, xy^kz is in L. Since |xy| <= N, y consists entirely of a's. Let |y| = m. By the lemma, the string xz is in L, and xz = $a^{N-m}ba^{2N}$. But 2N != 2(N-m), since m > 0, so this is a contradiction. Therefore, this language cannot be regular.

2. Let L = $\{ wa^{|w|} \mid w \in \{a,b,c\}^* \}$. In other words, L consists of words wa^n where w contains a's, b's , and c's and n is the length of w.

   i. Use the Myhill-Nerode Theorem to prove that L is not regular.

      Consider the sequence of strings b, bb, bbb, ..., b^k, ... for all k > 0. Pick any two of them, say b^k and b^m, where k != m. Then the word b^ka^k is in L but b^ma^k is not in L. Therefore, no two of the words in this infinite sequence are in the same equivalence class, proving that L must have an infinite number of such classes. Therefore, L is not regular, by the Myhill-Nerode theorem.

   ii. Use the Pumping Lemma to prove L is not regular.

      Let N be the constant of the lemma. Let w be the word b^Na^N. w is in L. By the lemma, there exist x, y, and z such that w = xyz, |xy| <= N, |y| > 0, and for all k, xy^kz is in L. Since |xy| <= N, y consists entirely of b's. Let |y| = m. By the lemma, the string xz is in L, and xz = $b^{N-m}a^N$. Since m > 0, this word cannot be in L, so this is a contradiction. Therefore, this language cannot be regular.

3. Let L = $\{ a^n \mid n \text{ is not a prime number} \}$.

   i. Prove that L is not regular.

      If L were regular, then its complement would be regular also, but the complement of L is the language we call PRIME, which we have already proved is not regular. Hence L is not regular.

   ii. Prove that L satisfies the Pumping Lemma.

      Let N = 6. For any word w in L whose length is at least 6, we can write w = xyz, where x is the null string, y = aa, and z is the rest of w. Note that |xy| <= 6 and |y| = 2 > 0. Because w is in L and its length is not a prime number, its length is an even number. Since |y| = 2, |xz| is an even number and cannot be 2, and for any k,

$|xy^kz| = |w| + 2k$ must also be an even number, implying it is not a prime number and hence xy^kz is in L.

4. Give an example of a regular language R and a non-regular language L such that R + L is regular, and prove or justify that R + L is regular.

   This is easy – let R be (a+b)* and let L be any of the non-regular languages above. The union of R and L is R, since R contains all languages over {a,b}.

5. Give an example of a regular language R and a non-regular language L such that R + L is non-regular, and prove or justify that R + L is non-regular.

   Let R be any finite language and let L be a language containing R that is not regular. Then R + L = L and L is not regular. As an example, let R = { a^2, a^3 } and let L = PRIME. PRIME contains R.

6. Let L be a regular language over Σ = {a,b}. Define L' = { x | there exists $y \in \Sigma^*$ such that $xy \in$ L }. Is L' regular? Either prove it is or give an example to show it may not always be.

   L' is regular. To see this, let M be a FA accepting L. Let it have states Q = { q_1, q_2, ..., q_n }. Let F be the set of final states of M. Let M' be a FA identical to M except for which states are final states. For each state q in Q for which there exists at least one word z such that $\delta^*(q, z)$ is a final state in M, make q a final state in M'.

   Let w be in L(M'). Then w reaches a final state of M', which means that w is a word that reaches a state q in M such that there is a word y such that $\delta^*(q, y)$ is in L. This implies that $wy \in \Sigma^*$ Therefore, w is in L'.

   Conversely, let w be in L'. Then there is a $y \in \Sigma^*$ such that wy is in L. Let q be the state in M that w reaches. Then $\delta^*(q, y)$ is a final state in M, which means that q is a final state in M', and hence w is in L(M'). This proves that M' accepts L'.