Problem 1: Round table
Consider $n$ people to be seated on a round table with $n$ chairs. How many possible seatings are there if:

(a) Two seatings are the same if every person has the same left neighbor and the same right neighbor.

Solution: We can assign the people to the chairs in $n!$ ways, as this is simply a permutation. However, in this case, the $n!$ overcounts the number of ways. To illustrate, consider an example where $n = 3$, and the people are $A$, $B$, and $C$. Take for instance the assignment of $A$ to the first chair, $B$ to the second, and $C$ to the third, which we can represent as $(A, B, C)$. Observe now that every rotation of this order is equivalent in terms of sitting on the round table, as each person preserves the left and right neighbors. Therefore, $(A, B, C)$, $(C, A, B)$, and $(B, C, A)$ are equivalent. We have 3 rotated seatings that are equivalent. In general, we have $n$. We conclude that every seating is counted $n$ times. So the number of ways we can seat people on a round table is $n!/n = (n-1)!$.

Another way to think about this problem is as follows: The first person will always be seated on the first chair, since the physical location is not important. This leaves us with $n-1$ people who can be permuted in $(n-1)!$ ways.

(b) Two seatings are the same if every person has the same set of two neighbors.

Solution: We can apply the same reasoning above. This time, however, we overcount even more. Let’s take the same example. Since the positioning of the left and right neighbor is irrelevant now, any reversal of the order will also be equivalent. Therefore, $(A, B, C)$, $(C, A, B)$, and $(B, C, A)$ are equivalent from before, and we add to that $(C, B, A)$, $(B, A, C)$, and $(A, C, B)$, by reversing each order respectively. So we generally overcount by $2n$. The number of ways is $(n-1)!/2$. Observe this works for $n \geq 3$, because when $n > 3$, reversal produces the same seating. So when $n > 3$, the number of ways is still $(n-1)!$.

Problem 2: Another sum
Assume that $n > 1$ is odd. Consider the following sum:

\[ \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{\frac{n-1}{2}} \]

(a) Write this sum using $\sum$ notation.
Solution:
\[ \sum_{k=0}^{(n-1)/2} \binom{n}{k} \]

(b) How many terms does this sum have?

Solution: We have terms that correspond to \( k = 0, \ldots, (n-1)/2 \). This is:
\[ \frac{n-1}{2} + 1 = \frac{n+1}{2} \]

In general, the number of integers in the set \( \{a, a+1, \ldots, b\} \) where \( a, b \in \mathbb{N} \), is \( b - a + 1 \).

(c) Show that the sum has an even number of terms that are odd.

Solution: Recall that \( n \) is odd. Looking at the odd rows of the Pascal triangle, we observe that each binomial coefficient appears exactly twice in a symmetric way. Stopping at \( k = (n-1)/2 \) is exactly the first half. So the above sum is equal to \( 2^{n/2} = 2^{n-1} \), which is even when \( n > 1 \). Now an even sum must have an even number of odd terms.

Problem 3: Traveling in Manhattan
Consider the following grid in Manhattan.

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An optimal path from point A to point B is a path that does not waste moves, i.e. it can only go up or to the right.

(a) Establish a one-to-one correspondence between optimal paths and binary patterns. What is special about these binary patterns.

Solution: Observe that we need exactly 8 steps to get from A to B on a path that does not waste any moves. Therefore, every path can be represented by a word of length 8 using the alphabet \( \{U, R\} \), where U stands for up and R stands for right. We can replace \( U \) with 1 and \( R \) with 0 to obtain a binary pattern that uniquely defines that path. Therefore, our function assigns to every path a binary pattern obtained by replacing \( U \) with 1 and \( R \) with 0. The set of binary patterns is precisely those that have 3 1s and 5 0s (this is how many up and right steps we need respectively). Our function is onto because every such binary pattern corresponds to a path. Moreover, two different paths will map to two different binary patterns. Therefore, our function is a one-to-one correspondence.

(b) Count the number of optimal path from A to B by counting the number of binary patterns above.
**Solution:** To count the number of paths, we need to count the number of binary patterns with 3 1s and 5 0s. To produce such a binary pattern, we need to choose 3 bits out of 8 and make them 1s. The rest of the bits are automatically 0s. Therefore, the number of such patterns is \( \binom{8}{3} \). Note: this is similar to the problem of rocks and partitions that we have seen in class: to partition \( k \) rocks into \( n \) groups we needed \( n - 1 \), and we transformed this into a binary pattern with \( k \) zeros and \( n - 1 \) ones, and we counted the number of those patterns as \( \binom{n + k - 1}{k} \).