Problem 1

Find the truth table for \( P \Rightarrow Q \) in the notes.

(a) Prove that
\[
(P \Rightarrow Q) \iff (\neg P \lor Q)
\]
- By case analysis of \( P \)
- Using a truth table

(b) Prove that for any two propositions \( P \) and \( Q \), the following is always true:
\[
(P \Rightarrow Q) \lor (Q \Rightarrow P)
\]
- By case analysis
- Using the above equivalence and the fact that associative property of \( \lor \), i.e. \( a \lor b \lor c = (a \lor b) \lor c = a \lor (b \lor c) \) (it does not matter where you put the parenthesis).

Problem 2: Proof by contradiction

Prove the following: There are no rational number solutions to the equation \( x^3 + x + 1 = 0 \), i.e. no solution can be written as a ratio \( a/b \) where \( a \) and \( b \) are integers (you can always consider \( a/b \) to be reduced to lowest terms). Hint: start your proof as you would start a proof by contradiction, then multiply by \( b^3 \) to get rid of the denominators. Then consider a case analysis of \( a \) and \( b \) based on even and odd.

Problem 3: Proof by contrapositive

Prove that if the product \( ab \) is irrational, then at least one of \( a \) or \( b \) is irrational.

Problem 4: Co-primes

(a) Show by case analysis that all integers from 0 to 7 can be constructed by using a linear combination of 3 and 8 (like we did with the water juggling puzzle).

(b) The greatest common divisor of 3 and 8 is 1, we call 3 and 8 co-primes. Because 3 and 8 are co-primes, we can represent all the integers as a linear
combination of 3 and 8. Can you show that when the greatest common divisor is not 1, not all numbers can be represented; for instance, consider 6 and 9? 

*Hint:* find a common factor of 6 and 9 and factor it out.

**Problem 5: A power set is uncountable**

(a) Use the diagonalization method to show that the set of all infinite binary sequences is uncountable.

(b) Show that the power set of a countable infinite set $S$ is uncountable. Recall that the power set of $S$, usually denoted by $2^S$ or $P(S)$, is the set of all subsets of $S$. 

*Hint:* Think of each subset of $S$ as an infinite binary sequence.