Problem 1

Find the truth table for $P \Rightarrow Q$ in the notes.

(a) Prove that

$$(P \Rightarrow Q) \iff (\neg P \lor Q)$$

• By case analysis of $P$
• Using a truth table

Solution:

Case analysis on $P$: If $P$ is false, then $P \Rightarrow Q$ is true, and so is $\neg P \lor Q$ because $\neg P$ is true and $\lor$ requires at least one true operand to be true. If $P$ is true, then we can observe from the truth table of $P \Rightarrow Q$, than $P \Rightarrow Q$ has the same truth value as $Q$, but that’s also the case of $\neg P \lor Q$ which is now $0 \lor Q$.

Truth table: We can easily construct the truth table for $\neg P \lor Q$ and observe that it is equivalent to $P \Rightarrow Q$.

(b) Prove that for any two propositions $P$ and $Q$, the following is always true:

$$(P \Rightarrow Q) \lor (Q \Rightarrow P)$$

• By case analysis
  • Using the above equivalence and the fact that associative property of $\lor$, i.e. $a \lor b \lor c = (a \lor b) \lor c = a \lor (b \lor c)$ (it does not matter where you put the parenthesis).

Solution:

Case analysis: If at least one of $P$ and $Q$ is false, then at least one of the implications is true, then the result is true. If, on the other hand, both $P$ and $Q$ are true, then both implications are true, and the result is true.

Using the equivalence: Since $(P \Rightarrow Q) \iff (\neg P \lor Q)$, then
\[(P \Rightarrow Q) \lor (Q \Rightarrow P) \iff [(\neg P \lor Q) \lor (\neg Q \lor P)]\]

but \((\neg P \lor Q) \lor (\neg Q \lor P) = \neg P \lor (Q \lor \neg Q) \lor P = \neg P \lor 1 \lor P = 1.\]

**Problem 2: Proof by contradiction**

Prove the following: There are no rational number solutions to the equation 
\[x^3 + x + 1 = 0,\] i.e. no solution can be written as a ratio \(a/b\) where \(a\) and \(b\) are integers (you can always consider \(a/b\) to be reduced to lowest terms). *Hint:* start your proof as you would start a proof by contradiction, then multiply by \(b^3\) to get rid of the denominators. Then consider a case analysis of \(a\) and \(b\) based on even and odd.

**Solution:** Assume that \(a/b\) is a solution, where \(a/b\) cannot be reduced. We have
\[
\frac{a^3}{b^3} + \frac{a}{b} + 1 = 0
\]
\[a^3 + ab^2 + b^3 = 0 \text{ (even)}\]

Case analysis:
- \(a\) odd, \(b\) odd: Then \(a^3 + ab^2 + b^3\) is odd, a contradiction.
- \(a\) odd, \(b\) even: Then \(a^3 + ab^2 + b^3\) is odd, a contradiction.
- \(a\) even, \(b\) odd: Then \(a^3 + ab^2 + b^3\) is odd, a contradiction.
- \(a\) even, \(b\) even: Then \(a/b\) can be reduced, a contradiction.

Therefore, the solution cannot be rational.

**Problem 3: Proof by contrapositive**

Prove that if the product \(ab\) is irrational, then at least one of \(a\) or \(b\) is irrational.

**Solution:** The contrapositive is the following: if \(a\) is rational and \(b\) is rational, then \(ab\) is rational. We can easily prove this: If \(a\) and \(b\) are both rational, then \(a\) can be written as \(x/y\) and \(b\) can be written as \(z/w\). Then \(ab\) is \((xz)/(yw)\), which is rational.

To elaborate on why the above is the contrapositive, let \(P\) be the proposition that \(ab\) is irrational, and \(Q\) be the proposition that \(a\) or \(b\) is irrational, which can be written as \(Q = A \lor B\), where \(A\) is the proposition that \(a\) is irrational and \(B\) is the proposition that \(b\) is irrational. It is clear that \(\neg P\) is the proposition that \(ab\) is rational. Now \(\neg Q\) is \(\neg(A \lor B)\). By DeMorgan’s law \(A \lor B = \neg(\neg A \land \neg B)\). Therefore, \(\neg Q = \neg A \land \neg B\). Therefore, \(\neg Q\) is the proposition that \(a\) and \(b\) are both rational.

**Problem 4: Co-primes**

(a) Show by case analysis that all integers from 0 to 7 can be constructed by using a linear combination of 3 and 8 (like we did with the water juggling puzzle).

**Solution:** Proof by case analysis:
\[
0 = 3(0) + 0(8)
\]
1 = 3(3) − 1(8)
2 = 6(3) − 2(8)
3 = 9(3) − 3(8)
4 = 12(3) − 4(8)
5 = 15(3) − 5(8)
6 = 18(3) − 6(8)
7 = 21(3) − 7(8)

(b) The greatest common divisor of 3 and 8 is 1, we call 3 and 8 co-primes. Because 3 and 8 are co-primes, we can represent all the integers as a linear combination of 3 and 8. Can you show that when the greatest common divisor is not 1, not all numbers can be represented; for instance, consider 6 and 9?

Hint: find a common factor of 6 and 9 and factor it out.

Solution: Assume we can write $x = 6a + 9b$. Then $x = 3(2a + 3b)$ because 3 is a common factor of 6 and 9. Therefore, $x$ must be a multiple of 3. So not every $x$ can be expressed this way.

Problem 5: A power set is uncountable
(a) Use the diagonalization method to show that the set of all infinite binary sequences is uncountable.

Solution: Assume the set is countable. Therefore, there is an order on all the infinite binary sequences. Let $s(i)$ be the $i^{\text{th}}$ sequence in that order. We construct an infinite binary sequence like this: The $i^{\text{th}}$ bit of this sequence is the opposite of the $i^{\text{th}}$ bit of $s(i)$. Obviously, such a sequence cannot be $s(i)$ for any finite $i$, a contradiction.

(b) Show that the power set of a countable infinite set $S$ is uncountable. Recall that the power set of $S$, usually denoted by $2^S$ or $P(S)$, is the set of all subsets of $S$.

Hint: Think of each subset of $S$ as an infinite binary sequence.

Solution: Since $S$ is countable, every element has a finite index in some order. So we can talk about the $i^{\text{th}}$ element. Given a subset of $S$, we can represent this subset by an infinite binary sequence like this: the $i^{\text{th}}$ bit of this sequence is 1 if and only if the subset contains the $i^{\text{th}}$ element. We thus establish a one-to-one correspondence between the $P(S)$ and the set of infinite binary sequences. Therefore, $P(S)$ must be uncountable since it has the same size as the set of all infinite binary sequences.