Problem 1: Program termination
Consider the following program:

\[(w, x, y, z) = (\{1, \ldots, n\}, \{1, \ldots, n\}, \{1, \ldots, n\}, \{1, \ldots, n\})\]
while \(w > 0\) and \(x > 0\) and \(y > 0\) and \(z > 0\)
control = \{1, 2, 3\}
if control == 1 then
    \(x = x, \ldots, n\)
    \(w = w - 1\)
else
    if control == 2 then
        \(y = y, \ldots, n\)
        \(x = x - 1\)
    else
        \(z = z, \ldots, n\)
        \(y = y - 1\)

Show that this program terminates using:

(a) Ramsey theory: Show that given any two iterations \(i < j\) (not necessarily consecutive), \(w_j < w_i\) or \(x_j < x_i\) or \(y_j < y_i\), and use the Ramsey argument about homogeneous sets, recall that this argument can be generalized to any number of colors.

(b) A partial order relation: construct a partial order relation on some tuple of the variables and show that for any two consecutive iterations \(i\) and \(i + 1\), the tuple at iteration \(i + 1\) comes before the tuple at iteration \(i\), i.e. tuples get “smaller”.

Problem 2: Proofs by induction

(a)
\[1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n + 1)^2}{4}\]

(b)
\[\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{n(n + 1)} = \frac{n}{n + 1}\]
(c) \[ 1 \cdot 2^0 + 2 \cdot 2^1 + 3 \cdot 2^2 + \ldots + n \cdot 2^{n-1} = (n - 1)2^n + 1 \]

(d) \( n^2 - 1 \) is a multiple of 4 if \( n \) is odd. **Hint:** your inductive step for \( n \) should consider \( n - 2 \) and not \( n - 1 \).

(e) Show that every positive integer can be written as the product of an odd number and a power of 2.