Problem 1: Program termination
Consider the following program:

\[(w, x, y, z) = \{1, \ldots, n\}, \{1, \ldots, n\}, \{1, \ldots, n\}, \{1, \ldots, n\}\]
while \(w > 0\) and \(x > 0\) and \(y > 0\) and \(z > 0\)
control = \{1, 2, 3\}
if control == 1 then
\[x = \{x, \ldots, n\}\]
\[w = w - 1\]
else
if control == 2 then
\[y = \{y, \ldots, n\}\]
\[x = x - 1\]
else
\[z = \{z, \ldots, n\}\]
\[y = y - 1\]

Show that this program terminates using:

(a) Ramsey theory: Show that given any two iterations \(i < j\) (not necessarily consecutive), \(w_j < w_i\) or \(x_j < x_i\) or \(y_j < y_i\), and use the Ramsey argument about homogeneous sets, recall that this argument can be generalized to any number of colors.

Solution: Between any two iterations \(i\) and \(j\), we consider three cases: (a) control is always 3, (b) control is sometimes 2 and sometimes 3, and (c) control is sometimes 1. In the first case, \(y\) decreases. In the second case, \(x\) decreases. In the third case, \(w\) decreases. The rest of the proof is similar to the one in the notes.

(b) A partial order relation: construct a partial order relation on some tuple of the variables and show that for any two consecutive iterations \(i\) and \(i + 1\), the tuple at iteration \(i + 1\) comes before the tuple at iteration \(i\), i.e. tuples get “smaller”.

Solution: Consider the tuple \((w, x, y)\) and define the partial order relation: \((w, x, y) \prec (w', x', y')\) iff \(w < w'\) or \(w = w'\) and \(x < x'\) or \(w = w'\) and \(x = x'\) and \(y < y'\). It is then obvious from the program that \((w_{i+1}, x_{i+1}, y_{i+1}) \prec (w_i, x_i, y_i)\)
\((w_i, x_i, y_i)\). Since we only have a finite number of possible tuples, this cannot happen indefinitely, so the program must stop.

**Problem 2: Proofs by induction**

(a)

\[
1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n + 1)^2}{4}
\]

**Solution:**

Base case: when \(n = 1\), this sum is 1, which is equal to \(\frac{1^2(1 + 1)^2}{4}\).

Inductive step: Assume the property is true for \(n - 1\); therefore,

\[
1^3 + 2^3 + \ldots + (n - 1)^3 = \frac{(n - 1)^2n^2}{4}
\]

Now for every \(n > 1\),

\[
\sum_{i=1}^{n} i^3 = \sum_{i=1}^{n-1} i^3 + n^3 = \frac{(n - 1)^2n^2}{4} + 4n^3 + \frac{n^2 + 2n}{4} = \frac{(n^2 + 1 - 2n)n^2 + 4n^3}{4}
\]

\[
= \frac{n^4 + n^2 + 2n^3}{4} = \frac{n^2(n^2 + 1 + 2n)}{4} = \frac{n^2(n + 1)^2}{4}
\]

(b)

\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{n(n + 1)} = \frac{n}{n + 1}
\]

**Solution:**

Base case: when \(n = 1\), we have \(\frac{1}{1 \cdot 2} = \frac{1}{1+1}\).

Inductive step: Assume the property holds for \(n - 1\); therefore,

\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{(n - 1)n} = \frac{n - 1}{n}
\]

For every \(n > 1\), the desired sum is

\[
\sum_{i=1}^{n-1} \frac{1}{i(i+1)} + \frac{1}{n(n + 1)} = \left[ \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{(n-1)n} \right] + \frac{1}{n(n + 1)}
\]

\[
= \frac{n - 1}{n} + \frac{1}{n(n + 1)} = \frac{(n - 1)(n + 1) + 1}{n(n + 1)} = \frac{n^2}{n(n + 1)} = \frac{n}{n + 1}
\]

(c)

\[
1 \cdot 2^0 + 2 \cdot 2^1 + 3 \cdot 2^2 + \ldots n \cdot 2^{n-1} = (n - 1)2^n + 1
\]

**Solution:**

Base case: when \(n = 1\), we have \(1 \cdot 2^0 = 1 = (1 - 1)2^1 + 1\).

Inductive step: Assume the property holds for \(n - 1\); therefore,

\[
1 \cdot 2^0 + 2 \cdot 2^1 + 3 \cdot 2^2 + \ldots (n - 1) \cdot 2^{n-2} = (n - 2)2^{n-1} + 1
\]

For every \(n > 1\), the desired sum can be written as

\[
\left[ 1 \cdot 2^0 + 2 \cdot 2^1 + 3 \cdot 2^2 + \ldots (n - 1) \cdot 2^{n-2} \right] + n \cdot 2^{n-1}
\]

\[
= (n - 2)2^{n-1} + 1 + n \cdot 2^{n-1} = (n - 2 + n)2^{n-1} + 1 = (2n - 2)2^{n-1} + 1 = 2(n - 1)2^{n-1} + 1 = (n - 1)2^n + 1
\]
(d) \( n^2 - 1 \) is a multiple of 4 if \( n \) is odd. Hint: your inductive step for \( n \) should consider \( n - 2 \) and not \( n - 1 \).

Solution:
Base case: when \( n = 1 \), \( 1^2 - 1 = 0 \) which is a multiple of 4.
Inductive step: Assume \( n \) is odd and the property holds for \( n - 2 \) (the previous odd). Then \((n - 2)^2 - 1 = 4k\) for some \( k \in \mathbb{N} \).
For every \( n > 1 \), let us write \( n = (n - 2) + 2 \). So

\[
(n - 2)^2 - 1 = \left[(n-2)+2\right]^2-1 = (n-2)^2+4+4(n-2)-1 = (n-2)^2-1+4[1+n-2] = 4k + 4k' \]

So \( n^2 - 1 \) is a multiple of 4.

(e) Show that every positive integer can be written as the product of an odd number and a power of 2.

Solution:
Base case: for \( n = 1 \), we have \( 1 = 1 \cdot 2^0 \).
Inductive step: Assume the property holds up to \( n - 1 \); therefore, all integers from 1 to \( n - 1 \) can be written as a product of an odd number and a power of 2. Given \( n > 1 \), we make a case analysis on \( n \).

\( n \) is odd: In this case \( n = n2^0 \), which is the product of an odd number and a power of 2.

\( n \) is even: In this case \( n = 2m \) where \( m < n \). Since \( m \leq n - 1 \), it can be written as the product of an odd number and a power of 2, say \( m = a2^k \) where \( a \) is odd, then \( n = a2^{k+1} \), which is also a product of an odd number and a power of 2.