1. In this exercise, we want to show that the interval \((-1, 1)\) is “as big” as the entire set \(\mathbb{R}\).

   - Find a function \(f : (-1, 1) \rightarrow \mathbb{R}\) that is both one-to-one and onto. This establishes the claim.

   - Consider the following geometric interpretation:

   > For any \(x \in (-1, 1)\), obtain its vertical projection on the unit circle in the upper half plane, then make an appropriate projection of that point onto the tangent line (the line \(y = 1\)), finally project vertically onto \(\mathbb{R}\) (the line \(y = 0\)). Show your work and explain.

Extra Challenge: (You are not required to do this) Find a bijection from \([-1, 1]\) to \(\mathbb{R}\).
2. We learned in class that for any set \( A \), \( |A| < |\mathcal{P}(A)| \). For instance, this means that \( \mathcal{P}(\mathbb{N}) \), the set of all subsets of \( \mathbb{N} \) is uncountable. Let \( \mathcal{P}_f(\mathbb{N}) \) be the set of all \textbf{finite} subsets of \( \mathbb{N} \).

\[
\mathcal{P}_f(\mathbb{N}) = \{ X \in \mathcal{P}(\mathbb{N}) \mid X \text{ is finite} \}
\]

For instance, the subset \( \{2, 4, 6\} \in \mathcal{P}_f(\mathbb{N}) \) but the subset \( \{1, 3, 5, \ldots\} \notin \mathcal{P}_f(\mathbb{N}) \). Show that \( \mathcal{P}_f(\mathbb{N}) \) is countable.

\textit{Note:} Here’s an ordering that does \textbf{not} work: order the elements of \( \mathcal{P}_f(\mathbb{N}) \) (finite subsets of \( \mathbb{N} \)) by their size (since each is finite, the size is well defined):

\[
\phi, \{1\}, \{2\}, \{3\}, \{4\}, \ldots
\]

Do you see the problem?

\textit{Hint:} Every finite subset has a largest element.

3. We will now show that \( \mathcal{P}(\mathbb{N}) \) is uncountable even though we already know this fact. This is an opportunity to practice the diagonal method. To do this, we will first represent each subset \( S \) of \( \mathbb{N} \) by an infinite binary word in which the \( i^{th} \) bit is 1 if \( i \in S \) and 0 otherwise. To practice this notion try to fill in the table:

<table>
<thead>
<tr>
<th>infinite binary word</th>
<th>subset of ( \mathbb{N} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>00000...</td>
<td></td>
</tr>
<tr>
<td>100000, ...</td>
<td></td>
</tr>
<tr>
<td>01110000...</td>
<td>{2, 4, 6}</td>
</tr>
<tr>
<td>1010101010...</td>
<td>{3k - 2 \mid k \in \mathbb{N}}</td>
</tr>
<tr>
<td></td>
<td>( \mathbb{N} )</td>
</tr>
</tbody>
</table>

Now we need to show that there is no bijection from \( \mathbb{N} \) to the set of infinite binary words. Reproduce the diagonal argument to show this fact.

\textit{Note:} Think about this but you are not required to provide an answer: Why doesn’t this diagonal argument disprove the fact that \( \mathcal{P}_f(\mathbb{N}) \) of question 2 is countable?

4. Let \( f : \mathbb{N} \to S \) be a function that is onto. Show that \( S \) is countable. \textit{Hint:} Find a subset of \( \mathbb{N} \) that has a bijection with \( S \).

5. Some birds can chirp and some birds can sing. We have 100 birds in total and we know that 46 of them can chirp. If only 17 birds are chirping singing birds, how many birds can sing? Find the answer using Venn diagram, then explain how you can apply the inclusion-exclusion principle to avoid the visual illustration.
6. How many numbers in \{1, 2, \ldots, 546\} are not divisible by 2 \textbf{and} not divisible by 3 \textbf{and} not divisible by 7?

\textit{Hint:} Negate the requirement, find the answer using inclusion-exclusion, then fix it to answer the original question.

\textit{Hint:} Check your answer against 546 \times (1/2) \times (2/3) \times (6/7) (this does not always work by the way depending on the choice of numbers).

\textbf{Problem}

How many 3 digit numbers do \textbf{not} have 1 in the first digit, \textbf{and} do \textbf{not} have 2 in the second digit, \textbf{and} do \textbf{not} have 3 in the third digit?

(a) Solve this question using the product rule by identifying the possibilities for each of the three digits.

(b) Do the same using the inclusion-exclusion principle. \textit{Hint:} consider the negation to get the “or” logic. In other words, if the above description defines what “good” 3 digit numbers are, find how many “bad” ones are there.

(c) Solve the question using the technique you used in (b) but with the added condition that all the 3 digits must be different.