Problem 1: An interesting language
Consider a language that uses the alphabet \{0, 1, #\}. In this language words obey one single rule: a # cannot follow a #. How many words of length \(n\) exist in this language? 

**Solution:** Let \(S_n\) be the number of words of length \(n\). If the word starts with a 0, we can finish it in \(S_{n-1}\) ways. Similarly, if the word starts with a 1, we can also finish it in \(S_{n-1}\) ways. Finally, if the words start with a #, then this must be followed by either a 0, and we can finish it in \(S_{n-2}\) ways, or a 1, and we can finish it in \(S_{n-2}\) ways as well. Since all these alternatives are exclusive, we can use the addition rule:

\[
S_n = 2S_{n-1} + 2S_{n-2}
\]

To solve this, we use the characteristic equation:

\[
x^2 = 2x + 2
\]

which gives the solutions:

\[
p = 1 + \sqrt{3} \quad q = 1 - \sqrt{3}
\]

We also know that \(S_0 = 1\) (empty word) and \(S_1 = 3\). So,

\[
S_0 = c_1 + c_2 = 1
\]

\[
S_1 = c_1(1 + \sqrt{3}) + c_2(1 - \sqrt{3}) = 3
\]

Solving we get:

\[
c_1 = (3 + 2\sqrt{3})/3 \quad c_2 = -(1 + 2\sqrt{3})/2
\]

Problem 2: A recurrence
Consider the following recurrence:

\[
a_n = a_{n-1} - a_{n-2}
\]

where \(a_0 = 0\) and \(a_1 = 1\).

(a) Using the recurrence and the the initial conditions, generate the first 18 numbers of the sequence \(\{a_n\}\). Try to guess a way to compute \(a_n\) immediately by simply knowing \(n\).
Solution:

\[ 0 \quad 1 \quad 1 \quad 0 \quad -1 \quad -1 \quad 0 \quad 1 \quad 1 \quad 0 \quad -1 \quad -1 \quad 0 \quad 1 \quad 1 \quad 0 \quad -1 \quad -1 \]

It is obvious that the sequence is periodic. We have

\[ a_n = \begin{cases} 
 0 & n \equiv 0, 3 \pmod{6} \\
 1 & n \equiv 1, 2 \pmod{6} \\
 -1 & n \equiv 4, 5 \pmod{6} 
\end{cases} \]

(b) Solve for \( a_n \). \textbf{Hint:} observe that \( a_n \) has the form \( a_n = Aa_{n-1} + Ba_{n-2} \), but you are going to encounter a little surprise!

\textbf{Solution:} Forming the characteristic equation, we have:

\[ x^2 = x - 1 \]

with two imaginary solutions

\[ p = \frac{1 + i\sqrt{3}}{2} \quad q = \frac{1 - i\sqrt{3}}{2} \]

Therefore,

\[ a_n = c_1 p^n + c_2 q^n \]

where

\[ a_0 = c_1 + c_2 = 0 \]
\[ a_1 = c_1 p + c_2 q = 1 \]

Solving for \( c_1 \) and \( c_2 \), we get:

\[ c_1 = -\frac{i}{\sqrt{3}} \quad c_2 = \frac{i}{\sqrt{3}} \]

(c) Your expression for \( a_n \) in part (b) most likely contains the imaginary number \( i \). Use the binomial theorem to obtain a nicer expression for \( a_n \):

\[ a_n = \frac{1}{2^{n-1}} \left[ \binom{n}{1} 3^0 - \binom{n}{3} 3^1 + \binom{n}{5} 3^2 - \ldots \right] \]

\textbf{Solution:}

\[ c_1 p^n = -\frac{i}{\sqrt{3}} \left( \frac{1}{2} + \frac{i\sqrt{3}}{2} \right)^n \]
\[ c_2 q^n = \frac{i}{\sqrt{3}} \left( \frac{1}{2} - \frac{i\sqrt{3}}{2} \right)^n \]

By expanding both expressions using the binomial theorem, the even terms cancel out and the odd terms add up. An odd term looks like \( (k \text{ is odd}) \)

\[ -\frac{i}{\sqrt{3}} \binom{n}{k} \left( \frac{1}{2} \right)^{n-k} \left( \frac{i\sqrt{3}}{2} \right)^k = i^{k+1} \binom{n}{k} \frac{1}{2^{n-1}} \sqrt{3}^{k-1} \]

When two of these terms add up, we get:

\[ -i^{k+1} \binom{n}{k} \frac{1}{2^{n-1}} \sqrt{3}^{k-1} \]

Now, \( i^{k+1} \) is an even power of \( i \), e.g. \( i^2, i^4, i^6, \ldots \), so these alternate signs, and \( \sqrt{3}^{k-1} \) is an even power of \( \sqrt{3} \), thus powers of 3. We get the desired result.