Problem 1: Truncated sum

(a) Check that
\[ \binom{7}{0} - \binom{7}{1} + \binom{7}{2} - \binom{7}{3} = -20 = -\binom{6}{3}. \]

(b) Generalize this result to
\[ \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots + \binom{n}{k} = (-1)^k \binom{n-1}{k}. \]

Solution:
\[ \binom{n-1}{0} - [\binom{n-1}{0} + \binom{n-1}{1}] + [\binom{n-1}{1} + \binom{n-1}{2}] - \ldots \\
\ldots + (-1)^k [\binom{n-1}{k-1} + \binom{n-1}{k}] \]

Terms cancel out, and the only one that remains is \((-1)^k \binom{n-1}{k}\).

(c) When \(|x| < 1\), \(1 - x + x^2 - x^3 + \ldots = (1 + x)^{-1}\). Give an alternative proof of (b) by comparing the coefficients of \(x^k\) on both sides of the identity:
\[ (1 + x)^{n-1} = (1 + x)^n (1 - x + x^2 - x^3 + \ldots) \]

_Hint:_ use the binomial theorem.

Solution: By the binomial theorem, the coefficient of \(x^k\) on the left hand side is given by \(\binom{n-1}{k}\). On the right hand side, many terms produce \(x^k\), we have:
\[ \binom{n}{n} x^n + \ldots + \binom{n}{k} x^k + \ldots + \binom{n}{0} x^0 (1 - x + x^2 - x^3 + \ldots) \]

The coefficient of \(x^k\) is:
\[ \binom{n}{k} - \binom{n}{k-1} + \ldots + (-1)^k \binom{n}{0} \]

So,
\[ \binom{n}{k} - \binom{n}{k-1} + \ldots + (-1)^k \binom{n}{0} = \binom{n-1}{k} \]
If we multiply both sides by \((-1)^k\), we get the result we want.

**Problem 2: A one-to-one correspondence**
Consider the function

\[ f : \mathbb{Z} \to \mathbb{N} \]

where

\[ f(x) = \begin{cases} 
2x & x \in \mathbb{N} \\
-2x + 1 & x \not\in \mathbb{N}
\end{cases} \]

(a) Show that \( f \) is a function.

**Solution:** For every \( x \in \mathbb{Z} \), either \( x \in \mathbb{N} \) or \( x \not\in \mathbb{N} \). Therefore, there is at most one element \( y \) in \( \mathbb{N} \) such that \( f(x) = y \). We need to show that this \( y \) is indeed in \( \mathbb{N} \). If \( x \in \mathbb{N} \), then \( y = 2x \) is also an element of \( \mathbb{N} \). If \( x \in \{0, -1, -2, -3, \ldots\} \), then \( y = -2x + 1 > 0 \), so \( y \in \mathbb{N} \).

(b) Show that this function is onto.

**Solution:** Consider \( y \in \mathbb{N} \). We need to show that there exists an \( x \in \mathbb{Z} \) such that \( f(x) = y \). If \( y \) is even, then let \( x = y/2 \). Observe that \( x \in \mathbb{Z} \) and \( x \in \mathbb{N} \). Therefore, \( f(x) = 2x = y \). If \( y \) is odd, then \( y = 2(k + 1) \) where \( k \in \{0, 1, 2, \ldots\} \). Therefore, \( y = -2(-k) + 1 \). Let \( x = -k \). Observe that \( x \in \mathbb{Z} \). Therefore, \( f(x) = -2(x + 1) = y \).

(c) Show that \( f \) is one-to-one, i.e. if \( x, y \in \mathbb{Z} \) and \( x \neq y \), then \( f(x) \neq f(y) \).

**Solution:** Consider \( x, x' \in \mathbb{Z} \). Assume \( f(x) = f(x') \) and so \( 2x = -2x' + 1 \). This means \( 2(x + x') = 1 \) which is impossible. So \( f(x) \) and \( f(x') \) must be different.

(d) What do we conclude about the two sets \( \mathbb{N} \) and \( \mathbb{Z} \).

**Solution:** Since \( f : \mathbb{Z} \to \mathbb{N} \) is one-to-one correspondence, \( \mathbb{Z} \) and \( \mathbb{N} \) have the same size.

**Problem 3: Integer solutions**
Consider the system:

\[ x_1 + x_2 + x_3 = 15 \]

(a) How many solutions exist if \( x_i \geq 0 \) for all \( i \)?

**Solution:** This is the number of ways to partition 15 among 3 parts: \( x_1 \), \( x_2 \), and \( x_3 \). This number is given by \( \binom{15 + 3 - 1}{3 - 1} \).

(b) How many solutions exist if \( x_i > 0 \) for all \( i \)? **Hint 1:** If \( x_1 > 0 \), then let \( x_1 = 1 + y_i \), where \( y_i \geq 0 \). Replace \( x_i \) by \( y_i \) and solve the system. **Hint 2:** This is similar to the case covered in the notes where every kid must receive at least one gift.

**Solution:** We can think of \( x_1 = 1 + x'_i \), where \( x'_i \geq 0 \). Replacing \( x_i \) by \( x'_i \), we have:

\[ x'_1 + 1 + x'_2 + 1 + x'_3 + 1 = 15 \]
\[ x'_1 + x'_2 + x'_3 = 12 \]
Therefore, the number of solutions to this system is \( \begin{pmatrix} 12 + 3 - 1 \\ 3 - 1 \end{pmatrix} \). Note that we can think of this transformation as a one-to-one function. Every solution \((x_1, x_2, x_3)\) corresponds uniquely to a solution \((x'_1, x'_2, x'_3)\), and vice-versa. Our function is \( f(x_1, x_2, x_3) = (x'_1, x'_2, x'_3) = (x_1 - 1, x_2 - 1, x_3 - 1) \).