Problem 1: Mouse trap
Recall the problem of the mouse trap, where we need to cover a grid that is
missing the bottom left and upper right corners with 2x1 traps. We proved by
contradiction that this cannot be done. First, we identified each square as being
even or odd, then argued that each trap must cover exactly one odd and one
even square. We concluded that the number of odd squares and that of even
squares must be equal, a contradiction (due to the two missing even corners).

One might ask the following: what if an area has the same number of odd and
even squares, can we always cover the area with traps? The answer is no. Find
a counter example.

Solution: Here’s a counter example: we have two even and two odd squares,
yet there is no way to cover this with traps.

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Problem 2: Sets
Show that \( A \cup B = (A^c \cap B^c)^c \)
(a) by picture.

(b) by showing that if \( x \in A \cup B \), then \( x \in (A^c \cap B^c)^c \) and vice-versa.

Solution: First we show that \( A \cup B \subset (A^c \cap B^c)^c \).

\[
\begin{align*}
x & \in (A \cup B) \\
x & \in A \text{ or } x \in B \\
x & \notin A^c \text{ or } x \notin B^c \\
x & \notin (A^c \cap B^c) \\
x & \in (A^c \cap B^c)^c
\end{align*}
\]
Then, we show that $(A^c \cap B^c)^c \subset A \cup B$.

$$x \in (A^c \cap B^c)^c$$

$$x \notin (A^c \cap B^c)$$

$$x \notin A^c \text{ or } x \notin B^c$$

$$x \in A \text{ or } x \in B$$

$$x \in (A \cup B)$$

**Problem 3: Contrapositive**

Given two propositions $A$ and $B$, show the following proposition is true:

(a) $(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$

**Solution:** Consider the truth table:

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
<th>$A \Rightarrow B$</th>
<th>$\neg B$</th>
<th>$\neg A$</th>
<th>$\neg B \Rightarrow \neg A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
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<td>1</td>
</tr>
</tbody>
</table>

(b) Use the above information to show the following proposition is true by using a truth table:

$r$ is irrational $\Rightarrow \sqrt{r}$ is irrational

by thinking of “$r$ is irrational” as proposition $A$ and “$\sqrt{r}$ is irrational” as proposition $B$, and using the equivalence. This is called proof by contrapositive.

**Solution:**

$$\sqrt{r} \text{ is rational } \Rightarrow \sqrt{r} = a/b \text{ where } a \text{ and } b \text{ are integers}$$

$$\sqrt{r} = a/b \Rightarrow r = a^2/b^2$$

$$r = a^2/b^2 \text{ where } a \text{ and } b \text{ are integers} \Rightarrow r \text{ is rational}$$

Therefore $r$ is irrational $\Rightarrow \sqrt{r}$ is irrational.
Problem 4: Binary words

Consider the infinite set $B$ of all binary words. This set contains words like

$$0$$

$$010$$

$$00$$

$$1001001$$

$$1010100$$

$$111110000001$$

Show that $B$ is countable. *Hint:* you cannot order these words based on their number representation because infinitely many words have the same representation; for instance, 0, 00, 000, 0000, 00000, ... are all the number 0.

**Solution:** We can order the words by length first, then within each length, we can order them by their number representation. Therefore, we have the following order:

$$0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111, 0000, \ldots$$

Observe that every word of length, say $l$, has a finite index in this order because there are finitely many words of length at most $l - 1$ and finitely many words of length $l$. In particular, there are $2^1 + 2^2 + \ldots + 2^{l-1}$ words of length at most $l - 1$. This is finite for a given $l$. 