Introduction to Stochastic Processes and Computer Simulation, CSc 85200 and STAT 702

Homework Assignment 3

Problem 1 Consider the model of a M/GI/1 queue with Poisson arrivals of rate λ (inter-arrival times are a sequence of iid exponential random variables with mean $1/\lambda$). Suppose that the service times $\{S_i\}$ are iid with general distribution. It is well known that if $\rho \stackrel{\text{def}}{=} \lambda \mathbb{E}(S_1) < 1$ then the stationary average queue length is:

$$\theta = \frac{\lambda^2 \, \mathbb{E}(S_1^2)}{2(1-\rho)}.$$

(a) Describe the discrete event model for the simulation: what are the variables in the physical state and what are the residual clocks?

(b) Write a code to estimate θ by simulating the queue length using the discrete event simulation model. Set $\lambda = 1$ and use the $\Gamma(3, 0.25)$ distribution for the service times. Notice that this is an example of a stationary average estimation. Explain how you determine the length of the simulation.

Problem 2 Using Little's Law for the M/GI/1 queue of the previous problem, the stationary average time in the system (waiting plus service) W satisfies $\theta = \lambda W$.

(a) Describe the Petri Net model for simulation using Lindley's equation: what is the state of the system? What is the natural filtration that makes the model a Markov process (with continuous state space)?

(b) Repeat (b) as in the previous problem, now using this simulation model instead.

Problem 3 Joey has r umbrellas that he uses in going from his home to his office, and vice versa. If he is at home (the office) at the beginning (end) of the day and it is raining, the he will take an umbrella with him to the office (home), provided there is one to be taken. If it is not raining, then he never takes an umbrella. Assume that, independent of the past, it rains at the beginning (end) of the day with probability p.

(a) Define a Markov Chain with $r + 1$ states, which will help determine the proportion of time that Joey gets wet.

(b) Let $q = 1 - p$. Show that the limiting probabilities are given by:

$$\pi_i = \begin{cases} \frac{q}{r+q} & \text{if } i = 0, \\ \frac{1}{r+q} & \text{if } i = 1, \ldots, r \end{cases} \tag{1}$$

(c) What fraction of the time is he going to get wet?

(d) When $r = 3$, what value of p maximizes the fraction of the time he gets wet?

Problem 4 Let $P^{(i)}$ be the transition matrix of an ergodic Markov chain with limit probabilities $\pi^{(i)}$, for $i = 1, 2$.

(a) Let $X_0 = 1$ be given. If the result of a coin toss is Heads (H) then $\{X_n\}$ will follow the dynamics given by $P^{(1)}$. Otherwise it will follow those given by $P^{(2)}$. Is $\{X_n\}$ a Markov chain? If so, determine the transition probabilities. Letting $p = \mathbb{P}(H)$, calculate the limiting probabilities: $\lim_{n\to\infty} \mathbb{P}(X_n = i)$?

(b) Now suppose that at each step we first throw a coin. If it results H then the following state is chosen according to $P^{(1)}$, otherwise it is chosen according to $P^{(2)}$. Is $\{X_n\}$ a Markov chain? If so, determine the transition probabilities. Show by counterexample that the limit probabilities are not the same as in (a) above.

Problem 5 Consider a population of individuals each of whom possesses two genes that can be either type A or type a. Suppose that in outward appearance (the *phenotype*) type A is dominant and type a recessive. Suppose that the population has stabilized, and the percentages of individuals having respective gene pairs AA, aa and Aa are p, q, and r Call an individual dominant or recessive depending on phenotype. Let S_{11} denote the probability that an offspring of two dominant parents will be recessive, S_{10} the probability that the offspring of one recessive and one dominant parent will be recessive. Compute S_{11} and S_{10} to show that $S_{11} = S_{10}^2$. These quantities are known in genetics as *Snyder's ratios*.

Problem 6 An algorithm is built to find the zeroes of a function. If this algorithm is in state j at the n-th step, then the proability of finding a zero in the next step is $1/j$. Otherwise its state will be $k \in \{1, 2, \ldots, j-1\}$ with probability $2k/j^2$. Find the mean number of iterations of the algorithm when we start at state m.

Please note that for problems 1 and 2 you are encouraged to work in teams to minimize the coding time. There are also resources available throughout the web and you may be able to copy/paste the basic simulation code or just use the basic recursive steps that I provided in pseudo-code in the class notes. The exercises are intended to help you understand the purpose of simulations when estimating stationary quantities.