<u>Game of Craps</u>. $\{Z_n\}$ outcome of 2dice (sums), iid. A

$n=0$: $Z_0 \in L \equiv \{2,3,12\} \Rightarrow$ game is lost

$Z_0 \in \{7,11\} \Rightarrow$ game is won

$Z_0 \in R \equiv \{4,5,6,8,9,10\} \Rightarrow$ continue game

Define

$$\tau = \min(n : Z_n \in \{Z_0, 7\})$$

The game is won when $Z_\tau = Z_0$ and lost when $Z_\tau = 7$.

We want to find $\mathbb{P}(\text{winning})$.

Let A denote the event of winning, that is:

$$A = \{\omega : Z_\tau = Z_0\}.$$

Because $n=0$ is different from the recursions for $n>0$, we can write down:

$$\mathbb{P}(A) = \mathbb{P}(\{7,11\}) + \sum_{Z_0 \in R} \mathbb{P}(A \mid Z_0)\mathbb{P}(Z_0)$$

$$= \frac{8}{36} + \sum_{Z_0 \in R} \mathbb{P}(A \mid Z_0)\mathbb{P}(Z_0) \quad \text{--- (0)}$$

We will now find a way to calculate $\mathbb{P}(A \mid Z_0)$, for any $Z_0 \in R$, using a change of measure that will make ~~winning~~ losing the game impossible.

New measure $\mathbb{Q}$:

$$\mathbb{Q}(Z_n = i) \equiv q(i) = \begin{cases} \frac{p(i)}{1-p(7)} & i \neq 7 \\ 0 & \text{otherwise} \end{cases}$$

so that the outcome $Z_n = 7$ (loss) has $\mathbb{Q}$-probability zero.

Here we have:

$$\tau = \min(n : Z_n = Z_0) \quad \text{a.s.-}\mathbb{Q}$$

So that τ has a geometric distribution:

$$\mathbb{Q}(\tau = n \mid Z_0) = q(Z_0)(1 - q(Z_0))^n \quad \text{---- (1)}$$

($= q(1-q)^n$ when no confusion arises).

[Ask students to show that τ is a stopping time w.r.t $\{\mathcal{F}_n\} = \{\sigma(Z_1, \ldots Z_n)\}$].

- Is $\mathbb{P} \ll \mathbb{Q}$?
- Is $\mathbb{Q} \ll \mathbb{P}$?

Consider the measure $\nu \equiv \mathbb{P}\rfloor_A$ on $\{2,3,4,5,6,8,9,10,11,12\}$ total mass is $1-p(7)$ (called a "defective probability").

Then $\nu \ll \mathbb{Q}$, and we can reproduce:

$$\mathbb{P}(A \mid Z_0) = \mathbb{E}_{\mathbb{Q}}\left(\prod_{n=1}^{\tau} \left[\frac{d\mathbb{P}}{d\mathbb{Q}}\right](Z_n) \underbrace{\mathbb{1}(Z_\tau = Z_0)}_{=1 \text{ under } \mathbb{Q}} \right)$$

$$\mathbb{P}(A \mid Z_0) = \mathbb{E}_{\mathbb{Q}}\left((1-p(7))^\tau \mid Z_0 \right)$$

Call $r \equiv 1 - p(7) \in (0,1)$.

Under $\mathbb{Q}$, given z_0, set $q = q(z_0)$, and τ has dist. (1), B $\Rightarrow$

$$\mathbb{E}_{\mathbb{Q}}(r^{\tau} \mid z_0) = \sum_{n=1}^{\infty} r^n q(1-q)^{n-1}$$

$$= r\sum_{n=1}^{\infty} q(r(1-q))^{n-1} \qquad (r(1-q) \in (0,1))$$

$$= \frac{qr}{1-r(1-q)} \underbrace{\sum_{n=1}^{\infty} [1-r(1-q)][r(1-q)]^{n-1}}_{\to 1}$$

$$\Rightarrow \mathbb{P}(A \mid z_0) = \frac{p(z_0)}{1-r+p(z_0)}, \text{ because } p(z_0) = rq(z_0) = (1-p(7))q(z_0)$$

Putting this result in (0), we obtain:

$$\mathbb{P}(A) = \frac{8}{36} + \sum_{z_0 \in R} \frac{p^2(z_0)}{p(z_0)+p(7)} = 0.49292929\ldots$$

Remark: Under $\mathbb{Q}$, the expected number of iterations to win the game is:

$$\mathbb{E}_{\mathbb{Q}}(\tau \mid z_0) = \frac{1}{q(z_0)} = \frac{1-p(7)}{p(z_0)}$$

So, if a simulation is performed starting with $z_0 \sim \mathbb{P}$, and then using $\mathbb{Q}$, the simulation length has expectation $6(1-p(7))$.