

CSCI 150 Fall 2026 TA's questions

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Week 1

Topics: The sum $T_n = 1 + 2 + \dots + n = n(n+1)/2$ (triangular numbers), planar graphs, Euler's formula for planar graphs $v - e + f = 2$, T_{n-1} is the number of pairs on n objects, generalization of sum to $a + (a+s) + (a+2s) + \dots + b = \frac{a+b}{2}(\frac{b-a}{s} + 1)$, counting pairs, permutations and $n!$, sum and product notations $\sum$ and $\prod$.

1. (Saad) Find a graph with 6 vertices and 9 edges that is not planar and a graph with 6 vertices and 9 edges that is planar. How many faces will such a graph have?
2. (Kacper) A construction company is building a theater with 50 rows of seats, where row n has n seats. They were able to complete the first 19 rows before having to purchase more seats. How many seats do they need to purchase to finish the remaining rows? (Solve without adding up the rows individually.)
3. (Zhen) A Pokemon can either have single or two different types. Let's say the developers of Pokemon decided to introduce 5 new types on top of the existing 18, totaling 23 different types. How many possible typings can a Pokemon have?
4. (Zhen) In a RPG game, a level 1 mob spawns with 25 health points (HP), and for each level increase, its HP increases by 3. In addition, if its level is greater 1, defeating it will drop its level by 1 while restoring full HP corresponding to that level. Only after defeating its level 1 will you beat the mob. You've encountered a level 31 mob, what's the minimum damage do you need to output in order to beat it?

5. (James) Imagine that a city planner is trying to design a new residential neighborhood with no bridges or tunnels. This neighborhood will feature exactly 17 blocks of road, think of these as street segments. The city planner wants these roads to connect in such a way that there are 12 pedestrian intersections. In this neighborhood, the city planner wants a school, a grocery store, a fire department, a bank, and three apartment complexes. The last part of the city planner's design is that each of these buildings will have no "neighbors", in other words each building is completely enclosed by roads. Can the architect's design work? Prove why or why not.
6. (Daniel) A basketball league has 30 teams, split evenly into two conferences of 15. Every team plays each of the other teams in its own conference exactly once, and plays each team in the opposing conference exactly once. How many games are played in a full season? Compute by separating the games played within conferences from the cross conference games, then verify your answer using a single calculation that ignores the conference structure. Both methods should produce the same number.
7. (Daniel) Evaluate

$$\sum_{k=3}^{20} (4k + 7)$$

using two different methods. First, split the sum using the properties of $\sum$ and express your work in terms of triangular numbers. Second, notice that consecutive terms differ by a fixed amount and apply the formula $a + (a + s) + (a + 2s) + \dots + b = \frac{a+b}{2}(\frac{b-a}{s} + 1)$. Confirm that both methods produce the same answer, and identify the value of s .

8. (Nathaniel) Rewrite the following using notation from the last few classes:

$$\prod_{k=4}^{17} (x)$$

$$\prod_{k=1}^x (k)$$

$$\sum_{k=1}^n (k)$$

Week 1 Solutions

1. (Saad) The graph $K_{3,3}$ (houses and utilities) has 6 vertices and 9 edges and is non-planar. Homework 0.5 has an example of a planar graph with 6 vertices and 9 edges; it must have 5 faces ($v - e + f = 2$ so $f = 2 - v + e = 2 - 6 + 9 = 5$).
2. (Kacper) Given that the company completed rows 1-19, and row n has n seats, the number of seats in the completed rows is $1 + 2 + 3 + \dots + 18 + 19$, which equates to

$$\sum_{k=1}^{19} (k) = \frac{19(20)}{2} = 190$$

Likewise, the number of remaining seats they have to complete is $20 + 21 + 22 + \dots + 49 + 50$, which equates to

$$\sum_{k=1}^{50} (k) - \sum_{k=1}^{19} (k) = \frac{50(51)}{2} - \frac{19(20)}{2} = 1275 - 190 = \boxed{1085}$$

3. (Zhen) A Pokemon can either be single or dual type. For single typing, there's 18 different ways. On the other hand there's $\binom{23}{2} = 253$ ways to assign dual typing. Adding them together yields $18 + 253 = 271$
4. (Zhen) At level k , the mob has $25 + 3(k - 1)$ HP. To beat it, we must defeat it once at every level from 31 down to 1. The HP values form an arithmetic sequence with first term $a = 25$, common difference $s = 3$, and last term

$$b = 25 + 30(3) = 115.$$

Using the arithmetic series formula, the minimum total damage is

$$\frac{a+b}{2} \left(\frac{b-a}{s} + 1 \right) = \frac{25+115}{2} \left(\frac{115-25}{3} + 1 \right) = 70(31) = \boxed{2170}.$$

5. (James) Translate the physical city into a graph. Because the architect cannot build bridges or tunnels, the graph must be planar.

Vertices (v): The 12 pedestrian intersections.

Edges (e): The 17 blocks of road connecting those intersections.

Faces (f): The 7 enclosed city blocks, plus the 1 outside region results in 8.

We need $v - e + f = 2$ for any planar graph, but $12 - 17 + 8 = 3$, so this graph is not planar and cannot be realized.

6. (Daniel) Within one conference, every pair of the 15 teams play once, so each conference has

$$\sum_{k=1}^{14} (k) = \frac{14(15)}{2} = 105$$

games, or 210 for both. Cross conference games pair each of the 15 teams on one side with each of the 15 on the other: $15 \times 15 = 225$. Total: $210 + 225 = 435$ To verify, forget the conferences. Every pair of the 30 teams plays once:

$$\sum_{k=1}^{29} (k) = \frac{29(30)}{2} = \boxed{435}$$

7. (Daniel) **Method 1:** split the sum and pull out constants:

$$\sum_{k=3}^{20} (4k+7) = 4 \sum_{k=3}^{20} (k) + 7(18) = 4 \left(\frac{20(21)}{2} - \frac{2(3)}{2} \right) + 126 = 4(207) + 126 = 954$$

(18 terms since $20 - 3 + 1 = 18$, and we subtract the sum to 2, not 3, to keep the $k = 3$ term.)

Method 2: first term 19, last term 87, step $s = 4$:

$$\frac{19 + 87}{2} \left(\frac{87 - 19}{4} + 1 \right) = 53 \times 18 = \boxed{954}$$

Both give 954 with $s = 4$.

8. (Nathaniel) The goal here is to see how these notations connect to other ones you've learned.

$$\prod_{k=4}^{17} (x) = x \cdot x \cdot x \cdot \dots \cdot x = x^{(17-4)} = x^{13}$$

$$\prod_{k=1}^x (k) = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (k-1) \cdot k = k!$$

$$\sum_{k=1}^n (k) = T(k)$$

Week 2

Topics: Manipulating sum and product notations. Snakes and ladders as a vehicle to explore: Abstraction, the Addition Rule, and the Multiplication Rule. Overcounting in Multiplication Rule.

1. (Kacper) A database system requires a 4-character administrator key with the following constraints

- Every character must be a hexadecimal digit (0-9 or uppercase A-F)
- The key must either:
 - Start with a letter and end with a digit
 - Start with a digit and end with a letter
- For security reasons, the key cannot have its middle two characters be the same (e.g., AXX0, A667)

Calculate the total number of valid administrator keys possible for this database system.

2. (Nathaniel)

ARMA VIRVMQVE CANO TROIAE QVI PRIMVS
AB ORIS

Much classical poetry, including this famous first line of Virgil's Aeneid, is written in Dactylic Hexameter. This meter is composed of six "feet", which may be dactyls (long-short-short syllables, -[˘][˘]) or spondees (long-long syllables, --). The first four feet can be either; the second to last foot must be a dactyl¹, and the last foot must be a spondee². For example, the above line scans to

Dactyl Dactyl Spondee Spondee **Dactyl Spondee**

-[˘][˘] -[˘][˘] -- -- -[˘][˘] --

How many valid lines of dactylic hexameter can you make? What if we want our line to have two or fewer dactyls?

¹Virgil actually does not strictly follow this rules, but violates it fewer than 30 times across his entire corpus

²Technically the last foot may be a spondee or trochee (-[˘]). For the sake of simplicity we will assume it is always a spondee

3. (Daniel) A candidate's onsite interview has 5 rounds: 3 coding, 1 system design round, and 1 behavioral round.
- If the coding rounds all have the same format, how many different orderings of the round types are possible?
 - What if each coding round is with a different interviewer, so all the rounds are distinguishable?
4. (Zhen) A piano has 88 keys, with 52 white keys and the rest black keys.
- (a) Bob wants to pick 10 unique keys for a music piece. How many ways can he choose?
 - (b) If he has to use equal amount of black and white keys for the 10 keys, how many ways can he choose?
 - (c) For the 10 keys that he chose, if he could only play each key once, how many ways can he play those keys?
5. (James) You are given a standard 52 card deck with 13 ranks: A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K; 2 colors: Red & Black; and 4 suits: hearts, diamonds, spades, clubs. The Red color consists of hearts & diamonds while the black color consists of spades & clubs.
- (a) How many ways can you select exactly 7 cards from the deck and then place them from left to right?
 - (b) How many ways can you select exactly 7 cards from the deck and then place them from left to right if we add the restriction that no two cards can be from the same rank. (In other words, if you select King of clubs, you can't select king of spades, king of diamonds, or king of hearts)
 - (c) How many ways can you select exactly 7 cards from the deck and then place from from left to right if we add the restriction that we select more black cards than red cards?

Week 2 Solutions

1. (Kacper) First, let's calculate the number of valid keys based on the two patterns:

- Pattern 1 (Starts with a letter and ends with a digit):
 - Position 1: 6 choices (A-F)
 - Position 2: 16 choices (0-9 + A-F)
 - Position 3: 16 choices (0-9 + A-F)
 - Position 4: 10 choices (0-9)
 - Total: $6 \times 16 \times 16 \times 10 = 15,360$
- Pattern 2 (Starts with a digit and ends with a letter):
 - Position 1: 10 choices (0-9)
 - Position 2: 16 choices (0-9 + A-F)
 - Position 3: 16 choices (0-9 + A-F)
 - Position 4: 6 choices (A-F)
 - Total: $10 \times 16 \times 16 \times 6 = 15,360$

Since a key can't simultaneously start with a letter and a digit, these two patterns are disjoint, therefore the number of valid raw keys is $15,360 + 15,360 = 30,720$

However, this sum includes keys with the same middle two characters, which aren't valid keys because of the security constraint. Therefore, we need to find out how many of such keys exist, and subtract them from the total pool of possible keys:

- Forbidden Keys in Pattern 1:
 - Position 1: 6 choices (A-F)
 - Position 2: 16 choices (0-9 + A-F)
 - Position 3: 1 choice (must match position 2)
 - Position 4: 10 choices (0-9)
 - Total: $6 \times 16 \times 1 \times 10 = 960$
- Pattern 2 (Starts with a digit and ends with a letter):
 - Position 1: 10 choices (0-9)
 - Position 2: 16 choices (0-9 + A-F)
 - Position 3: 1 choice (must match position 2)

- Position 4: 6 choices (A-F)
- Total: $10 \times 16 \times 1 \times 6 = 960$

Therefore, the final sum for the number of valid keys in this database system is

$$30,720 - (960 + 960) = 30,720 - 1,920 = \boxed{28,800}$$

- (Nathaniel) We can apply the product rule here. The last two feet must be a dactyl and spondee, and therefore are the same for every valid line and can be ignored. We then see that each position has two options, and apply the product rule to get

$$2 \cdot 2 \cdot 2 \cdot 2 = 2^4$$

valid lines. Remember that lines with the same number of feet in different orders are different lines!

For the second part, we can apply the addition rule. Since a line cannot have only one dactyl and exactly two dactyls at the same time, we know that we are looking at disjoint sets. Now we must add how many combinations there are for each. Because every valid line must have a dactyl in the 5th foot, we know that the only line with exactly one dactyl must have spondees everywhere else. This gives 1 valid line with 1 dactyl. For two dactyls, we now have a dactyl to place somewhere in the first four feet. Because only one of these can be a dactyl and all the rest must be spondees, we simply care about how many places we can place our dactyl: 4.

We now apply the addition rule, since we have disjoint sets that cover all of the options. $1 + 4 = 5$ Valid dactylic hexameter lines with two or fewer dactyls.

- (Daniel) If all 5 rounds were different, there would be $5! = 120$ orderings. But the 3 coding rounds are identical, so swapping them gives the same schedule. Each schedule gets counted $3! = 6$ times:

$$\frac{5!}{3!} = \frac{120}{6} = \boxed{20}$$

To check, place the unique rounds directly: 5 slots for system design, then 4 for behavioral, and coding fills the rest: $5 \times 4 = 20$.

If the coding rounds are with different interviewers, all 5 rounds are distinguishable, so nothing is overcounted: $5! = \boxed{120}$.

4. (Zhen)

- (a) $\binom{88}{10}$
- (b) $\binom{52}{5} \binom{88-52}{5}$
- (c) $10!$

5. (James)

- (a) Using the product rule, we can think about selecting 7 cards first in $\binom{52}{7}$ ways, then permuting them to get an order from left to right in $7!$ ways:

$$\binom{52}{7} \cdot 7!$$

Observe that this is exactly

$$\frac{52!}{(52-7)!}$$

because we select 7 card with order and no repetition. This example shows the relation between $\binom{n}{k}$ and $\frac{n!}{(n-k)!}$. There is a $k!$ factor difference.

- (b) Using the product rule again: First select the 7 ranks out of 13 you want to use. Then for each of these 7 ranks select the suit (there are 4 ways to select each suit leading to 4^7 ways for this phase). Finally permute the 7 cards you have.

$$\binom{13}{7} \cdot 4^7 \cdot 7!$$

- (c) To have more black cards than red cards in a 7 card selection, the hand must contain some number k of black cards where k is 4, 5, 6, or 7. For the different values of k , those categories are disjoint. Therefore, we can count each category, then use the addition rule. For any fixed k , let us focus on the product rule. If we have k black cards, then we must have $7 - k$ red cards, then we must permute all 7. There are $\binom{26}{k} \binom{26}{7-k}$ ways of choosing k black cards then choosing $7 - k$ red cards. We can then permute the 7 cards in $7!$ ways, leading to $\binom{26}{k} \binom{26}{7-k} 7!$ ways. Finally, by the addition rule, we must have

$$\sum_{k=4}^7 \binom{26}{k} \binom{26}{7-k} \cdot 7!$$

total number of ways for choosing more black than red cards.

Week 3

Topics: The 4 ways of making selections (we established formulas for all except the case of unordered selection with repetition), introduction to sets, counting subsets, and functions, onto, one-to-one, bijection.

1. (Saad) For any given positive integer n , consider the set $S = \{1, 2, 3, \dots, 2n\}$.

- Let T be the set of all tuples $(i, j) \in S^2$ such that $i < j$, and i and j are even. Write T in set notation and find $|T|$.
- Let R be the set of all tuples $(i, j) \in S^2$ such that $i < j$, i is even, and j is odd. Write R in set notation and find $|R|$ using the following helper function:

$$f : T \rightarrow R$$

$$f(i, j) = (i, j - 1)$$

2. (Daniel) Wordle answers are 5 letter strings. Ignore whether a string is an actual English word, so any sequence of 5 letters counts.

- How many 5 letter strings are possible from the 26 letter alphabet?
- Starting with a guess where all 5 letters are different is generally better. How many strings have 5 different letters?
- A word like ROBOT has exactly one repeated letter and its other three letters are all different. How many strings look like this? Then check whether your answer to the last two parts add up to the first, and explain what you find.

3. (James) In computer systems "virtual memory" refers to a technique that provides abstraction between a programmer and the actual physical memory resources on a computer. Let V be the set of virtual pages, let F be the set of physical frames on a computer, and let f be the function mapping from pages to physical frames. Assume that every frame is being used.

$$f : V \rightarrow F$$

$$|V| = n$$

$$|F| = m$$

$$n > m$$

- Can f be one-to-one? Explain why or why not. (Later, we formalize this idea as the Pigeonhole principle.)
- What is the largest possible number of pages that can be mapped to one individual frame? What is the smallest?
- Imagine that the computer designers are interested in reverse engineering what pages mapped to a frame, given a specific frame in the hardware. Consider the reverse mapping G

$$g : F \rightarrow V$$

if

$$f(x) = y$$

then

$$g(y) = x$$

where x is a page in the set V and y is a frame in the set F . Categorize g . Can it be a function? If so can it be onto or one-to-one?

- For the following exercise, work with the given specified values. There are 11 virtual pages and 4 frames. We know that 5 pages map to the first frame, 2 pages map to the second frame, 2 pages map to the third frame, and 2 pages map to the 4th frame. Under these circumstances, how many functions f are possible?

Week 3 Solutions

1. (Saad)

•

$$T = \{(i, j) \in S^2 \mid i < j \text{ and both are even}\}$$

In order to construct a tuple (i, j) we simply select two even integers from S . The selection process cannot have repetition since $i < j$. In addition, since the order is imposed in the tuple, the order by which we make the selection is irrelevant. This is $\binom{n}{2}$ since we have n even integers in S . Here's another way of seeing this through the multiplication rule:

$$\begin{array}{rcl} \text{choose an even integer} & \dots\dots\dots & n \\ \text{choose another even integer} & \dots\dots & n-1 \\ & & \text{-----} \\ & & n(n-1) \end{array}$$

There is the standard overcount by 2 since order is not relevant, so we get $n(n-1)/2$, which is $\binom{n}{2}$. Another way to think about this is by observing that either (i, j) or (j, i) is not valid. (This is another example where one needs to be careful about the wording, since the mention of order in the setting does not necessarily imply ordered selection.)

•

$$R = \{(i, j) \in S^2 \mid i < j \text{ and } i \text{ is even and } j \text{ is odd}\}$$

The multiplication rule now becomes tricky, since if we choose an even integer, we must then choose an odd integer that is larger. This does not translate well to using the multiplication rule, as the number of ways of carrying out the second stage is no longer well determined (the second stage is not independent from the choices made in the first). Instead, let us show that the function f is a bijection.

First, f is one-to-one:

$$\begin{aligned} f(i, j) = f(i', j') &\Rightarrow (i, j-1) = (i', j'-1) \Rightarrow i = i' \text{ and } j-1 = j'-1 \\ &\Rightarrow i = i' \text{ and } j = j' \Rightarrow (i, j) = (i', j') \end{aligned}$$

Second, f is onto: Consider $(i, j) \in R$ (where $i < j$ and i is even and j is odd). By definition of f , $f(i, j+1) = (i, j)$. Now, let's see if $(i, j+1) \in T$. Observe that $i < j+1$. Furthermore, and since j is odd, it must be that $j+1$ is even. In addition, for every odd integer $x \in S$, $x+1$ is also in S , so $j+1$ does not exceed the range. So $(i, j+1) \in T$.

Since f is both one-to-one and onto, it is a bijection. Therefore, $|R| = |T| = \binom{n}{2}$.

2. (Daniel) Each of the 5 positions independently take any of 26 letters, an ordered selection with repetition:

$$26^5 = \boxed{11,881,376}$$

For 5 different letters, each position uses up a letter, so this is an ordered selection without repetition:

$$26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 = \boxed{7,893,600}$$

For strings like ROBOT, build the string in three stages. First pick the repeated letter: 26 ways. Then pick which 2 of the 5 positions it occupies: $\binom{5}{2} = 10$ ways. Note this is not $5 \cdot 4 = 20$, since both positions hold the same letter and ordering them overcounts by $2!$. Finally the other 3 positions must all differ from each other and from the repeated letter, leaving $25 \cdot 24 \cdot 23 = 13,800$ ways. By the product rule:

$$26 \cdot 10 \cdot 13,800 = \boxed{3,588,000}$$

Checking whether these cover everything:

$$7,893,600 + 3,588,000 = 11,481,600 \neq 11,881,376$$

The two cases are disjoint but not exhaustive, missing 399,776 strings with heavier repetition: two separate pairs (LEVEL), one letter three times (ERROR), a triple and a pair, or four and five of a kind. The addition rule needs cases that are both disjoint and cover every possibility.

3. (James)

- Since every frame in the co-domain F is used, f must be an onto function. If it is one-to-one as well, then it's a bijection, implying that $|V| = |F|$. But $n > m$ so that's impossible. Therefore, f cannot be one-to-one.

- The smallest number of pages that can be mapped to an individual frame is $\boxed{1}$, since we know f is an onto function. For the largest, it is $\boxed{n - m + 1}$. This is because we know that at minimum every frame has one paged map to it, and there are m frames. So at most, there are $n - m$ pages leftover as extra once we account for that. If all of these extra pages mapped to the same frame, we get $n - m$ plus the one minimum resulting in $n - m + 1$.
- We know that the reverse mapping g is not a function; for instance, it is possible that $f(v_1) = f(v_2) = w$ where w is some frame in F . In this case, $g(w)$ cannot be defined.

- $\binom{11}{5} \binom{6}{2} \binom{4}{2} \binom{2}{2}$

$$= \frac{11!}{5!2!2!2!}$$

Choose 5 pages out of the original 11 to map to the first frame. Then choose 2 pages of the remaining 6 to map to the second frame. Then choose 2 pages from the remaining 4 to map to the third frame. Finally, map the last 2 pages to the final frame.

Week 4

All 4 variations of selecting k out of n . Integer solutions to $x_1 + \dots + x_n = k$.
Binomial coefficients, binomial theorem, and anagrams.

1. (Nathaniel) You are a game designer working on the original Halo Combat Evolved. Your job is to place enemies on the map for the player to fight. A Covenant squad consists of:
 - 2-6 Grunts
 - 0-3 Jackals
 - 1-4 Elites
 - (a) How many different possible squads are there altogether?
 - (b) The map designer has decided he wants a squad of 12 enemies. How many are there?
 - (c) Now playtesters are trying to determine how combat flows against different enemy types. How many ways can you place the squads from part (b) in order of when the player encounters each enemy? (Assume enemies of the same type are interchangeable)
 - (d) In the original game, Grunts can have ranks Minor, Major, or Spec Ops; Jackals can have ranks Minor or Major, and Elites can be Minor, Major, Stealth, Spec Ops, or Commander. Pick a squad using the rules from (a) and determine how many different combinations of enemy ranks you can make.
2. (Kacper) A university database contains 12 distinct student records. A query retrieves exactly 4 different records and displays them as a list.

While testing the application, a developer notices that if the same four records appear in a different order, the application treats the result as a new output. The developer then modifies the application so that rearranging the records no longer produces a different output.

 - (a) How many different results can the modified query display compared to the original query?
 - (b) Of the 12 records, 5 belong to Computer Science majors, and 7 belong to students in other majors. How many different results can the modified query display if at least 3 of the records belong to Computer Science majors.

- (c) To answer part (b), a system administrator says, "Pick 3 CS majors in $\binom{5}{3} = 10$ ways, then pick any 1 of the remaining 9 students, giving $10 \times 9 = 90$ ways." What is wrong with the system administrator's approach?
3. (Daniel) Hunter requires you to take exactly 15 credits next semester, and every course is worth 3 credits, so you need exactly 5 courses. You have narrowed your choice down to four categories: CS elective, math requirements, gen eds, and free electives. You do not care which specific course you pick inside a category, only how many courses you take from each.
- (a) How many ways can you split your 5 courses across the four categories, if a category may receive zero courses?
- (b) Your advisor tells you that you must take at least one course from every category. Now how many splits are possible?
- (c) The CS department changes the rule: you must take at least 2 CS electives, but the other three categories have no minimum. How many splits are possible now? Explain how you handled the minimum without listing cases, and explain why this answer is larger than your answer to part (b).

Week 4 Solutions

1. (Nathaniel)

- (a) This can be thought of similar to stars and bars. Let's say we choose grunts first: we are essentially choosing to place a "stop" somewhere between 2 and 6 grunts (or after 6), for $\binom{6-1}{1} = 5$ options to choose grunts. Because the other placements are independent, we can do the same things and apply the product rule. $\binom{4}{1} = 4$ options to choose jackals. $\binom{4}{1} = 4$ options to choose elites. for $5 \cdot 4 \cdot 4 = 80$ ways to create a squad.
- (b) Because we need 12 enemies to complete this squad, we simply choose one of the 13 possible enemies to leave out, either a Grunt, a Jackal, or an Elite. This gives 3 squads.
- (c) You can think of this as identical to the anagram example on the slides. For each of the three possible squads we have an anagram: GGGGGGJJJEEE, GGGGGGJJEEEE, GGGGGJJJEEEE. Since these are disjoint cases, we can solve them separately and sum up the result:

$$\frac{12!}{(6! \cdot 3! \cdot 3!)} + \frac{12!}{(6! \cdot 2! \cdot 4!)} + \frac{12!}{(5! \cdot 3! \cdot 4!)}$$

- (d) We can use stars and bars here, because only the number of soldiers with a given rank is relevant (and not which particular soldier has the rank). For example, for the squad with 4 Grunts, 2 Jackals, and 3 Elites, we first place two divisions in the Grunts (switching from placing Minors to Majors and Majors to Spec Ops) for $\binom{4+2}{2} = 15$ ways to choose ranks for our Grunts. For example, two Minors, a Major, and two Spec Ops would be

$$**|*|*$$

For our Jackals we only need to place one bar in three stars and bars for 3 possibilities. Finally our Elites place four in seven for $\binom{7}{4} = 35$. Finally, we apply the multiplication rule for $15 \cdot 3 \cdot 35 = 1,575$ ways to choose our squad's ranks. The other configurations are handled similarly.

2. (Kacper)

$$(a) P(12, 4) = \frac{12!}{8!} = \boxed{11\,880}, \text{ vs } \binom{12}{4} = \frac{12!}{4!8!} = \boxed{495}$$

- (b) Since we have 4 CS Majors, "at least 3" out of 4 means exactly 3 or exactly 4. These are disjoint cases we can add up:

$$\text{Exactly 3 CS: } \binom{5}{3} \binom{7}{1} = 10 \times 7 = 70$$

$$\text{Exactly 4 CS: } \binom{5}{4} \binom{7}{0} = 5 \times 1 = 5$$

$$70 + 5 = \boxed{75}$$

- (c) The issue with the system administrator's approach is that "the remaining 9 students" includes the 2 leftover CS majors that were not picked. This means that the same result can be counted more than once. For example, suppose CS Majors A, B, C, D are all selected. The sysadmin's method produces this result in four different ways:

$$\{A, B, C\} + D, \{A, B, D\} + C, \{A, C, D\} + B, \{B, C, D\} + A$$

3. (Daniel) Let x_1, x_2, x_3, x_4 be the number of courses taken from each category, so we are counting nonnegative integer solutions to

$$x_1 + x_2 + x_3 + x_4 = 5$$

With no restrictions this is stars and bars with 5 stars and 3 dividers:

$$\binom{\underbrace{4}_n + \underbrace{5}_k - 1}{3} = \binom{8}{3} = \boxed{56}$$

If every category needs at least one course, give one course to each category up front. That uses 4 of the 5 courses and leaves 1 to distribute freely:

$$\binom{\underbrace{4}_n + \underbrace{1}_k - 1}{3} = \binom{4}{3} = \boxed{4}$$

If only CS has a minimum of 2, assign those 2 up front, leaving 3 courses to distribute freely among all four categories:

$$\binom{\underbrace{4}_n + \underbrace{3}_k - 1}{3} = \binom{6}{3} = \boxed{20}$$

In both restricted cases the technique is substitution: let $y_i = x_i - (\text{minimum for category } i)$, which turns a problem with minimums into

one without, then count with stars and bars. Part (c) gives a larger answer than part (b) even though its minimum is bigger, because what limits the count is how many courses are left over after the minimums are paid. Part (b) spends 4 courses on minimums and leaves only 1, while part (c) spends 2 and leaves 3.

Week 5

Logic, AND $\wedge$, OR $\vee$, NOT $\neg$, Imply (if then) $\Rightarrow$, iff $\Leftrightarrow$, truth tables, Boolean functions, proofs: case analysis, direct, by contradiction, existential...

1. (Saad) Consider this statement: "If P , then if Q , then R ". Is it equivalent to "If P and Q , then R "? Show they are equivalent by writing them using $\Rightarrow$, then replacing $\Rightarrow$ with other logical operators.
2. (Saad) Let x and y be bits, and consider $x + y$. As in base 10, adding the two bits, may result in a carry. Give two Boolean functions using the operators $\wedge$, $\vee$, and $\neg$ to find $x + y$, and their carry c .
3. (Saad) Prove the following is false:

$$\forall x \in \mathbb{N}, x \text{ is prime} \Rightarrow x \text{ is odd}$$

in two ways:

- by providing a counter example
- by negating it and providing an example (thus showing that the negation is true). Observe that the negation of $\forall x, P(x)$ is $\exists x, \neg P(x)$. Here, $P(x)$ is: $x \text{ is prime} \Rightarrow x \text{ is odd}$. So recall how to negate something of the form $P \Rightarrow Q$.

Week 5 Solutions

- (Saad) First, "If P , then if Q , then R " can be written as $P \Rightarrow (Q \Rightarrow R)$. This is equivalent to $\neg P \vee (\neg Q \vee R) = \neg P \vee \neg Q \vee R$. Now consider "If P and Q , then R ", which can be written as $P \wedge Q \Rightarrow R$. This is equivalent to $\neg(P \wedge Q) \vee R$. By DeMorgan's law, this is $(\neg P \vee \neg Q) \vee R = \neg P \vee \neg Q \vee R$.

- Here's a truth table for $x + y$ and c :

x	y	$x + y$	c
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

Using the method illustrated in class:

$$x + y = (\neg x \wedge y) \vee (x \wedge \neg y)$$

$$c = x \wedge y$$

- (Saad) For the statement to be true, every integer in $\mathbb{N}$ must make this true:

$$x \text{ is prime} \Rightarrow x \text{ is odd}$$

Here's a counter example when $x = 2$: (2 is prime $\Rightarrow 2$ is odd) has the form $True \Rightarrow False$, which is false.

If we negate the statement, we get:

$$\exists x \in \mathbb{N}, x \text{ is prime} \wedge \neg(x \text{ is odd})$$

here we change $\forall$ to $\exists$, and we negate $P \Rightarrow Q$ by using $P \wedge \neg Q$. Obviously, $x = 2$ is an example that makes this statement true: It is prime, and it is not odd.

Week 6

Proofs in general...

1. (Saad) Prove by contradiction that a triangular number cannot be a positive power of 2. *Hint:* if $ab = 2^x$ then both a and b are powers of 2.
2. (Saad) Consider the set $S = \{1, 2, \dots, 100\}$. Prove by contradiction that there is no $x \in S$ such that $2x$ is equal to the sum of all other integers in S .
3. (Saad) Use the contrapositive to prove the following, where x and y are integers:
$$x + y \text{ is odd} \Rightarrow x \neq y$$
4. (Saad) Prove by case analysis (try all) that if n is not a multiple of 3, then n^2 is not a multiple of 3. *Hint:* consider all cases of the remainder in the division of n by 3. Can you state the contrapositive?
5. (Saad) Prove that there exists a Fibonacci number greater than 1 that is a square.

Week 6 Solutions

1. (Saad) Assume that there is a triangular number that is a positive power of 2; for instance, $T_n = 2^k$, where $k > 0$ then:

$$T_n = \frac{n(n+1)}{2} = 2^k \Rightarrow n(n+1) = 2^{k+1}$$

This means both n and $n+1$ are powers of 2, and they are both even (none of them can be 1 since $1 \cdot 2 = 2^1$ and $k+1$ is at least 2. But one of them must be odd, a contradiction.

2. (Saad) Let's assume that there is $x \in S$, such that $2x$ is equal to the sum of all other integers in S . This means that the sum of all integers in S must be equal to $3x$ and, therefore, $100 \times 101/2 = 3x$. So $5050 = 3x$ and $x = 5050/3$, a contradiction since 5050 is not divisible by 3.
3. (Saad) The contrapositive is:

$$x = y \Rightarrow x + y \text{ is even}$$

Here's a proof:

$$x = y \Rightarrow x + y = x + x = 2x \Rightarrow x + y \text{ is even.}$$

4. There are only two cases to consider for n not a multiple of 3:

- Remainder of 1: $n = 3k + 1 \Rightarrow n^2 = (3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1 = 3k' + 1$
- Remainder of 2: $n = 3k + 2 \Rightarrow n^2 = (3k + 2)^2 = 9k^2 + 12k + 4 = 9k^2 + 12k + 3 + 1 = 3(3k^2 + 4k + 1) + 1 = 3k' + 1$

The contrapositive: n^2 is a multiple of 3 $\Rightarrow n$ is a multiple of 3. Observe that we can strengthen this to iff:

$$n \text{ is a multiple of } 3 \Leftrightarrow n^2 \text{ is a multiple of } 3$$

This is because if n is a multiple of 3, then n^2 is also a multiple of 3 ($n = 3k \Rightarrow n^2 = 9k^2 = 3(3k^2)$).

5. (Saad) Here's the Fibonacci sequence:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots$$

Observe that 144 is 12^2 .