

Boolean function

$$f : \{0,1\}^n \rightarrow \{0,1\}$$

$$\{0,1\}^n = \underbrace{\{0,1\} \times \{0,1\} \times \dots \times \{0,1\}}_{n \text{ times}}$$

Example: $f : \{0,1\}^3 \rightarrow \{0,1\}$

x	y	z	$f(x,y,z)$
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

Any Boolean function can be constructed using

$\{\neg, \vee, \wedge\}$ operators

We say $\{\neg, \vee, \wedge\}$ is UNIVERSAL

$$f(x,y,z) = (\neg x \wedge \neg y \wedge z) \vee (\neg x \wedge y \wedge \neg z) \\ \vee (x \wedge \neg y \wedge \neg z) \vee (x \wedge y \wedge z)$$

x	y	z	$f(x, y, z)$
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

$x \quad y \quad z$

0 0 1

$\overline{x \wedge y \wedge z}$

0 1 0

$\overline{x \wedge y \wedge z}$

1 0 0

$\overline{x \wedge y \wedge z}$

1 1 1

$\overline{x \wedge y \wedge z}$

✓

$\{\neg, \wedge\}$ is universal

$\{\neg, \vee\}$ is universal

De Morgan's Law:

$$\neg(A \wedge B) = \neg A \vee \neg B$$

$$\neg(A \vee B) = \neg A \wedge \neg B$$

} proof? Truth Table

[Example: $\overbrace{a \notin \mathbb{Q}}^A \vee \overbrace{b \notin \mathbb{Q}}^B$

Negate above statement : $a \in \mathbb{Q} \wedge b \in \mathbb{Q}$

Commutative:

$$A \wedge B = B \wedge A$$

$$A \vee B = B \vee A$$

Associative:

$$A \wedge (B \wedge C) = (A \wedge B) \wedge C = A \wedge B \wedge C$$

$$A \vee (B \vee C) = (A \vee B) \vee C = A \vee B \vee C$$

Distributive:

$$\overbrace{A \wedge (B \vee C)} = (A \wedge B) \vee (A \wedge C)$$

$$\overbrace{A \vee (B \wedge C)} = (A \vee B) \wedge (A \vee C)$$

Properties of Implication

P	Q	$P \Rightarrow Q$
0	0	1
0	1	1
1	0	0
1	1	1

1) P is true and $(P \Rightarrow Q)$ is true, then Q is true.
(look at last row) [direct proof]

2) $(\neg Q \Rightarrow \neg P) = (P \Rightarrow Q)$ [proof by contrapositive]

3) $(\neg P \Rightarrow \text{false})$ is true, then P is true
[proof by contradiction]

(look at row in which Q is False but $P \Rightarrow Q$ is true) (1st row)

4) $(P \Rightarrow Q)$ is true and $(Q \Rightarrow R)$ is true
then $(P \Rightarrow R)$ is true. [transitive]

Let's prove property (4):

by case analysis of P.

P is False : $(P \Rightarrow R)$ is true regardless of R

P is True : P is true and $(P \Rightarrow Q)$ is true

then Q is true.

Q is true and $(Q \Rightarrow R)$ is true , then

R is true.

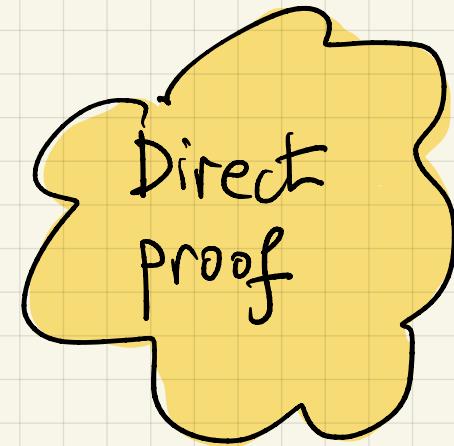
Therefore $(P \Rightarrow R)$ is true

Example 1 : Prove $\underbrace{n \text{ odd}}_P \Rightarrow \underbrace{n^2 \text{ is odd}}_Q$ (is true)

Definition:

$$n \text{ odd} : n = 2 \cdot k + 1, k \in \mathbb{Z}$$

$$n \text{ even} : n = 2 \cdot k, k \in \mathbb{Z}$$



$$\underbrace{P: n \text{ is odd}}_{\text{def.}} \Rightarrow n = 2k + 1, k \in \mathbb{Z}$$

$$\begin{aligned} n = 2k + 1 &\Rightarrow n^2 = (2k+1)^2 = 4k^2 + 4k + 1 \\ &= 2(\underbrace{2k^2 + 2k}_{k' \in \mathbb{Z}}) + 1 \end{aligned}$$

$$n^2 = 2k' + 1 \Rightarrow \underbrace{n^2 \text{ is odd.}}_{\text{def.}}$$

We also proved: $n^2 \text{ is even} \Rightarrow n \text{ is even.}$

Example 2: Prove $\underbrace{ab \notin \mathbb{Q}}_{A} \implies \underbrace{(a \notin \mathbb{Q} \vee b \notin \mathbb{Q})}_{B}$

Consider the contrapositive:

$$\neg B \implies \neg A$$

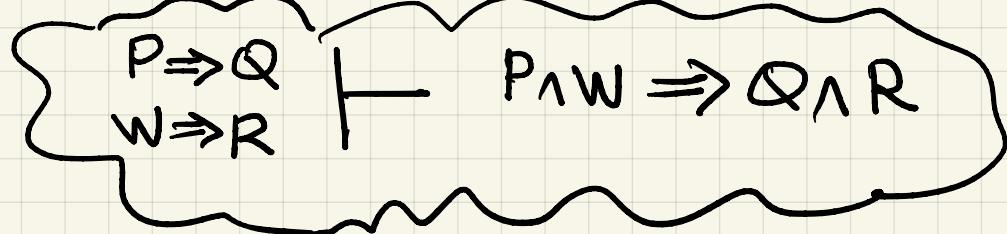
$$a \in \mathbb{Q} \wedge b \in \mathbb{Q} \implies ab \in \mathbb{Q}$$



$$\bullet a \in \mathbb{Q} \implies a = \frac{x}{y}, x \in \mathbb{Z} \text{ and } y \in \mathbb{N}$$

$$\bullet b \in \mathbb{Q} \implies b = \frac{z}{w}, z \in \mathbb{Z} \text{ and } w \in \mathbb{N}$$

$$\underbrace{a \in \mathbb{Q} \wedge b \in \mathbb{Q}}_{\text{Hypothesis}} \implies a = \frac{x}{y} \text{ and } b = \frac{z}{w}$$



$$a = \frac{x}{y} \quad \text{and} \quad b = \frac{z}{w} \Rightarrow$$

$$ab = \frac{x}{y} \cdot \frac{z}{w} = \frac{x \cdot z}{y \cdot w}, \quad x, z \in \mathbb{Z}, y, w \in \mathbb{N}$$

$$\Rightarrow \underbrace{ab}_{\in \mathbb{Q}}$$

Example:

Prove $\sqrt{2}$ is irrational.

by contradiction

P: $\sqrt{2} \notin \mathbb{Q}$

$\neg P$: $\sqrt{2} \in \mathbb{Q}$

$\sqrt{2} \in \mathbb{Q} \Rightarrow \sqrt{2} = \frac{a}{b}$ (a & b are integers and $\frac{a}{b}$ is irreducible)

$\Rightarrow 2 = \frac{a^2}{b^2} \Rightarrow a^2 = 2b^2 \Rightarrow a^2$ is even $\Rightarrow a$ is even.

$\sqrt{2} \in \mathbb{Q} \Rightarrow 2 = \frac{a^2}{b^2} \Rightarrow b^2 = \frac{a^2}{2} = \frac{2k \times 2k}{2} = \frac{4k^2}{2} = 2k^2$

$\Rightarrow b^2$ is even $\Rightarrow b$ is even

a is even and b is even $\Rightarrow \frac{a}{b}$ is reducible

$\sqrt{2} \in \mathbb{Q} \Rightarrow (\frac{a}{b}$ is reducible and $\frac{a}{b}$ is irreducible)
False