

## Example functions

$$f: \mathbb{Z} \rightarrow \mathbb{N}$$
$$f(x) = |x| + 1$$

$$g: \mathbb{N} \rightarrow \mathbb{Z}$$
$$g(x) = 2x$$

$$h: \mathbb{N} \rightarrow \{2, 4, 6, 8, \dots\}$$
$$h(x) = 2x$$

$$w: \mathbb{N}^2 \rightarrow \mathbb{N}$$
$$w(x, y) = x+y$$

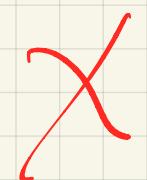
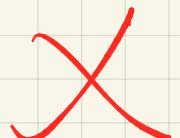
onto



one-to-one



bijection



# Worksheet

$$f: \mathbb{Z} \rightarrow \mathbb{N}$$

$$f(x) = |x| + 1$$

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$$g(x) = 2x$$

$$h: \mathbb{N} \rightarrow \{2, 4, 6, 8, \dots\}$$

$$h(x) = 2x$$

$$w: \mathbb{N}^2 \rightarrow \mathbb{N}$$

$$w(x, y) = x + y$$

onto

one-to-one

bijection

for  $y \in \mathbb{N}$ , let  $x = y - 1 \geq 0$  ✓  $f(-1) = f(1)$

$$f(x) = |x| + 1 = x + 1 = y - 1 + 1 = y$$

No  $x \in \mathbb{N}$  such that ✗  $f(x_1) = f(x_2) \Rightarrow$   
 $f(x) = -1$  ✓  $x_1 = x_2$

For  $y \in \{2, 4, \dots\}$ , let  $x = \frac{y}{2} \in \mathbb{N}$ , (same above)  
 then  $f(x) = 2x = 2 \frac{y}{2} = y$ .

$w(x, y) \geq 2$ , No  $(x, y)$  ✗  
 such that  $w(x, y) = 1$ . ✗  $f(x, y) = f(y, x)$

## Binary Patterns

- How many binary words with 10 bits can we make?

$$2^{10} \quad (\text{why?})$$

- How many binary words with 10 bits and 3 1s can we make?

$$\binom{10}{3} \quad (\text{why?})$$

- Same as above but consecutive 1s must be separated by at least two 0s.

$$\begin{array}{ccccccc} & \underbrace{1} & \underbrace{1} & \underbrace{1} & & \\ x_1 & & x_2 & & x_3 & & x_4 \\ \geq 0 & & \geq 2 & & \geq 2 & & \geq 0 \end{array}$$

$$x_1 + (2+x_2) + (2+x_3) + x_4 = 7$$

$$x_1 + x_2' + x_3' + x_4 = 3, \quad x_1, x_2', x_3', x_4 \geq 0$$

$\nwarrow_n \swarrow_k$

$$\binom{n-k+1}{n-1}$$

$$\binom{4-1+3}{4-1} = \binom{6}{3}$$

## The binomial coefficients

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1) \dots (n-k+1)}{k!}$$

Some properties

[symmetry]  $\binom{n}{k} = \binom{n}{n-k}$  e.g.  $\binom{5}{3} = \binom{5}{2}$

[Pascal triangle]  $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ ,  $0 < k < n$

e.g.  $\binom{5}{3} = \binom{4}{2} + \binom{4}{3}$

# Algebraic Proofs

$$\begin{aligned} \bullet \quad \binom{n}{k} &= \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)! k!} = \frac{n!}{(n-k)![n-(n-k)]!} = \binom{n}{n-k} \\ \bullet \quad \binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-1-k)!} \\ &= \frac{k(n-1)! + (n-k)(n-1)!}{k!(n-k)!} \\ &= \frac{(n-1)!(k+n-k)}{k!(n-k)!} = \frac{(n-1)! n}{k!(n-k)!} = \frac{n!}{k!(n-k)!} \\ &= \binom{n}{k} \end{aligned}$$

# Pascal Triangle

Row

0 ....

1

1 ....

1 1

2 ....

1 2 1

....

1 3 3 1

....

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

$$(x+y)^0 = 1$$

$$(x+y)^1 = 1 \cdot x + 1 \cdot y$$

$$(x+y)^2 = 1 \cdot x^2 + 2 \cdot xy + 1 \cdot y^2$$

$$(x+y)^3 = 1 \cdot x^3 + 3 \cdot x^2y + 3 \cdot xy^2 + 1 \cdot y^3$$

Let  $P(n, k)$  be  $k^{\text{th}}$  number in row  $n$  (both  $n$  &  $k$  start at 0)

$P(n, k) = P(n-1, k-1) + P(n-1, k)$  (does it remind you of something?)

It turns out  $P(n, k) = \binom{n}{k}$

- Why are they called Binomial coefficients ?
  - They are the coefficients of  $x^{n-k}y^k$  in the expansion of the binomial  $(x+y)^n$

- **Binomial theorem :**

$$\begin{aligned}
 (x+y)^n &= \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k \\
 &= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \dots + \binom{n}{n} y^n
 \end{aligned}$$