More examples of recurrences

$$R_n = R_{n-1} + n$$

$R_0 = 1$, $R_1 = 2$, $R_2 = 4$

$$R_n = R_{n-1} + n$$

$$\underline{R_{n-1} = R_{n-2} + (n-1)}$$

$$R_n - R_{n-1} = R_{n-1} + n - R_{n-2} - (n-1)$$

$$R_n - R_{n-1} = R_{n-1} - R_{n-2} + 1$$

$$R_n = 2R_{n-1} - R_{n-2} + 1$$

$$R_n = 2R_{n-1} - R_{n-2} + 1$$

$$R_{n-1} = 2R_{n-2} - R_{n-3} + 1$$

$$R_n - R_{n-1} = 2R_{n-1} - R_{n-2} - 2R_{n-2} + R_{n-3} + 1 - 1$$

$$\boxed{R_n = 3R_{n-1} - 3R_{n-2} + R_{n-3}}$$

$$x^3 = 3x^2 - 3x + 1$$

$$\boxed{x^3 - 3x^2 + 3x - 1 = 0} \rightarrow p, q, r$$

$$(x-1)^3 = 0$$

$$(x-1)(x^2 + ax + 1) = 0$$

$$-x^2 + ax^2 = (a-1)x^2$$

$$a - 1 = -3$$

$$a = -2$$

$$(x-1)(x^2 - 2x + 1) = 0$$

$$(x-1)^2$$

$$p = q = r = 1$$

$$R_n = c_1 p^n + c_2 n p^n + c_3 n^2 p^n = p^n [c_1 + c_2 n + c_3 n^2]$$

$$= c_1 + c_2 n + c_3 n^2$$

$$R_0 = 1 = c_1 + c_2 \times 0 + c_3 \times 0^2 = c_1$$

$$R_1 = 2 = c_1 + c_2 \times 1 + c_3 \times 1^2 = 1 + c_2 + c_3$$

$$\boxed{c_2 + c_3 = 1}$$

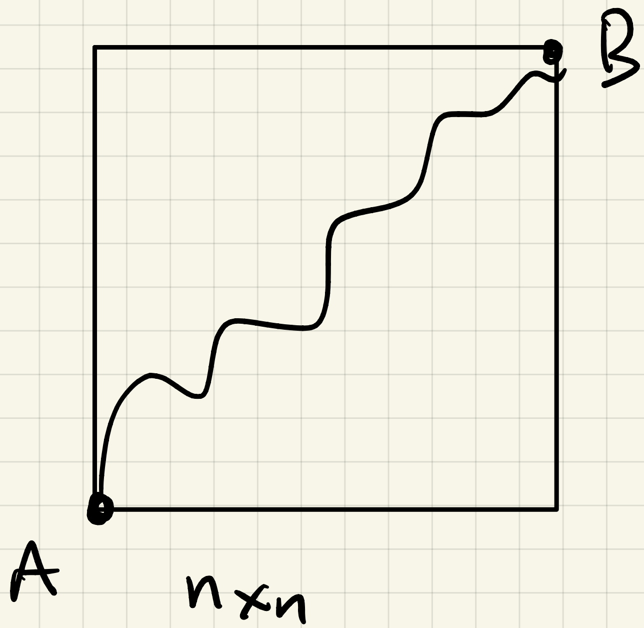
$$R_2 = 4 = c_1 + c_2 \times 2 + c_3 \times 2^2$$

$$= 1 + 2c_2 + 4c_3 = 4$$

$$c_1 = c_2 = \frac{1}{2}$$

$$R_n = 1 + \frac{1}{2} n + \frac{1}{2} n^2 = 1 + \frac{1}{2} n(n+1)$$

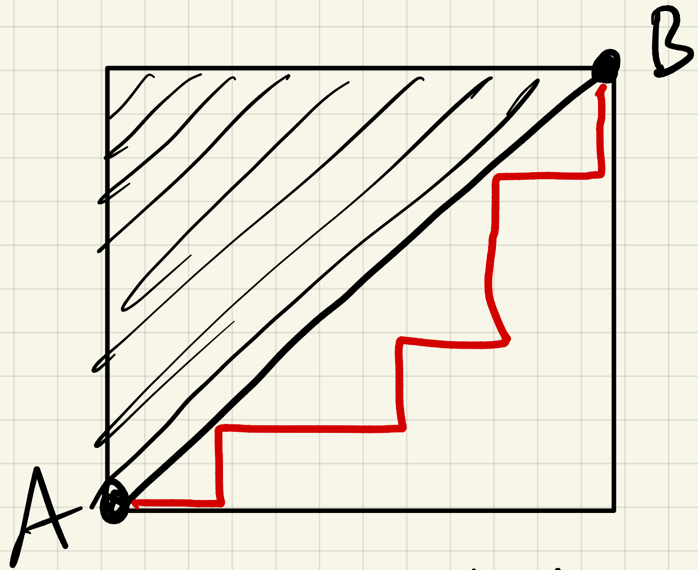
Catalan Numbers



Using only Up and Right moves

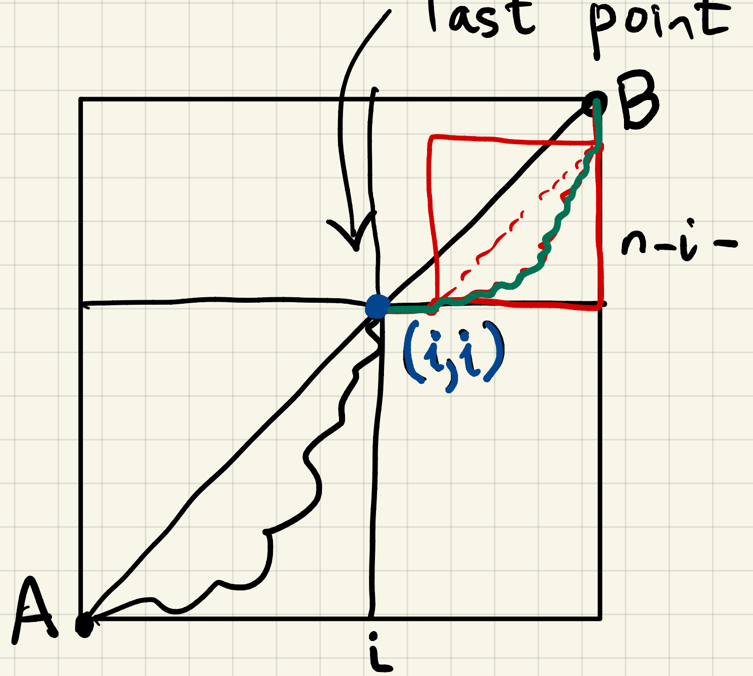
$$\# \text{path} = \binom{2n}{n}$$

What if we cannot cross the diagonal?



how many Up/Right paths are there ?

last point where the path touches the diagonal



Go from A to (i,i) then from (i,i) to B

(product rule)

sum over all i

$$C_n = \sum_{i=0}^{n-1} C_i \cdot C_{n-i-1}$$

$$C_n = \sum_{i=0}^{n-1} C_i \, C_{n-i-1}$$

Catalan number recurrence.

$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-1} C_0$$

$C_0 = 1$

$C_1 = C_0 C_0 = 1$

$C_2 = C_0 C_1 + C_1 C_0 = 2$

$C_3 = C_0 C_2 + C_1 C_1 + C_2 C_0 = 2 + 1 + 2 = 5$

1, 1, 2, 5, 14, 42, 139, 429, ...

It turns out:

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Generating function:

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$f(x)^2 = (c_0x^0 + c_1x^1 + c_2x^2 + \dots)(c_0x^0 + c_1x^1 + c_2x^2 + \dots)$$

$$= \sum_{n=0}^{\infty} (c_0c_n + c_1c_{n-1} + \dots + c_nc_0)x^n$$

$$= \sum_{n=0}^{\infty} c_{n+1}x^n = \frac{1}{x}\sum_{n=0}^{\infty} c_{n+1}x^{n+1} = \frac{1}{x}[f(x) - 1]$$

$$x f(x)^2 - f(x) + 1 = 0$$

$$f(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

when $x = 0$, $f(x) = c_0 = 1$. So we must choose $f(x) = \frac{1}{2x}\left[1 - (1-4x)^{\frac{1}{2}}\right]$

Expanding $(1-4x)^{\frac{1}{2}}$ as $\sum a_n x^n$ reveals that coefficient of x_n is $\frac{1}{n+1}\binom{2n}{n}$

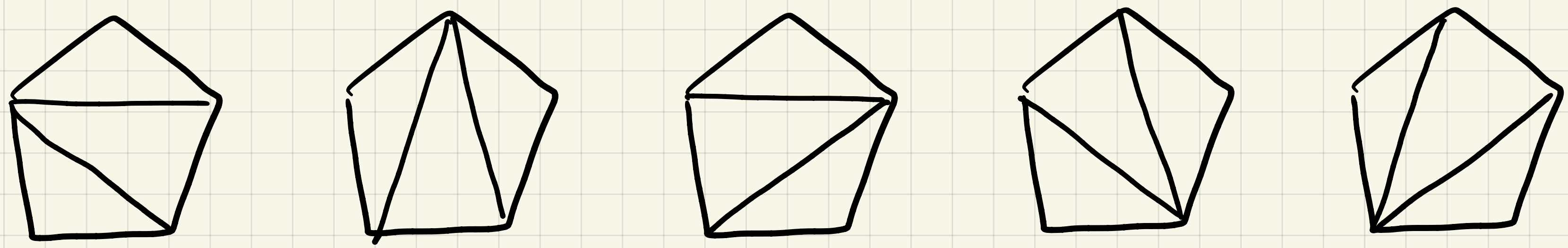
$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Parenthesis : n pairs $(n=3)$

$$()()(),\ ()(()),\ (())(),\ (()()),\ ((()))$$

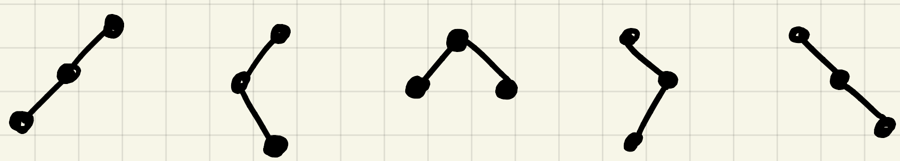
$$P_n = C_n$$

Triangulation : n sides $(n=5)$



$$T_n = C_{n-2}$$

Binary Trees : n nodes $(n=3)$



$$T_n = C_n$$