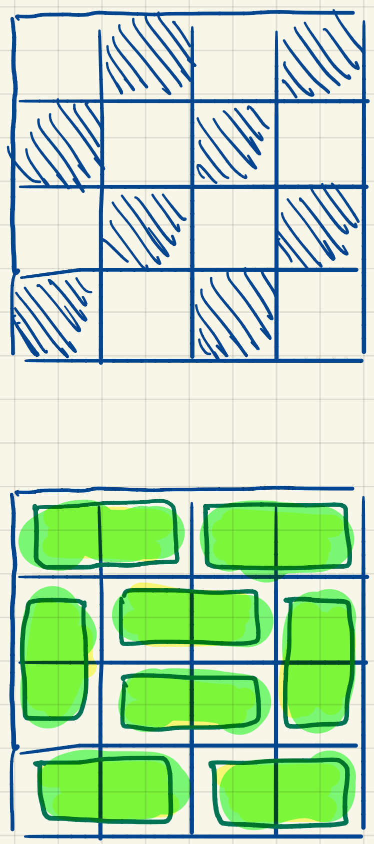
Guess

 $P \Rightarrow Q$

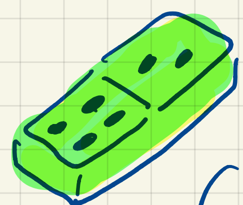
answer: Water proof

lecture 11

Example 4: Consider the $n \times n$ (n even)

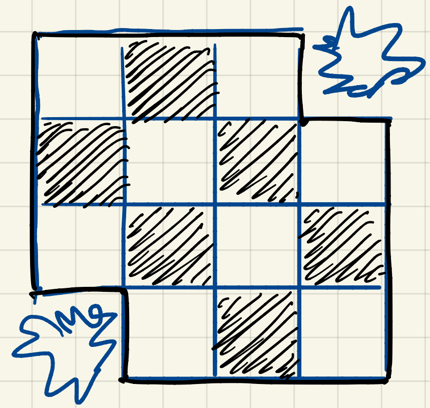


Cover it by dominos.



(all dominos must lie exactly in the grid and occupy two cells)

Show: $\underbrace{\text{opposite corners deleted}}_{P} \Longrightarrow \underbrace{\text{grid not coverable}}_{Q}$



Suppose opposite corners deleted $\wedge$ coverable (is true)

$P \wedge \neg Q$

$P \Rightarrow$ two cells of same color deleted $\Rightarrow$

$\#\text{black} \neq \#\text{white}$

$\neg Q \Rightarrow$ each domino covers exactly one black cell and one white cell $\Rightarrow$

$\#\text{black} = \#\text{white}$

$$P \wedge \neg Q \Rightarrow \underbrace{(\#b \neq \#w \ \wedge \ \#b = \#w)}_{\text{False}} \qquad \text{(contradiction)}$$

so $P \Rightarrow Q$ is true.

Primes are infinite

Consider the set of prime numbers

$$\mathbb{P} = \{2, 3, 5, 7, \ldots\} = \{p_1 = 2,\ p_2 = 3,\ p_3 = 5, \ldots\}$$

Prove $\mathbb{P}$ is infinite.

$\mathbb{P}$ finite $\Rightarrow$ $\exists\, k \in \mathbb{N}.$ p_k is the largest prime $\Rightarrow$

$\exists\, n \in \mathbb{N}.\ n = 1 + \prod_{i=1}^{k} p_i = p_1 \times p_2 \times p_3 \times \cdots \times p_k + 1 \Rightarrow$

n is not divisible by any prime $\wedge$ $n > p_k \Rightarrow$

n is prime $\wedge$ $n > p_k$ (Contradiction)

There is no smallest positive rational

Let $x > 0$ be smallest positive rational

$x \in \mathbb{Q} \Rightarrow x = \frac{a}{b} \wedge a \in \mathbb{Z} \wedge b \in \mathbb{N} \Rightarrow$

$\exists\, y \in \mathbb{Q}.\ y = \frac{x}{2} = \frac{a}{2b} \Rightarrow y \in \mathbb{Q} \wedge y < x.$

(contradiction)

(Note: $x \neq 0$)

$\frac{x}{2} < x$

Back to direct Proofs

Generalizing even/odd results:

Show n even $\Longrightarrow n^k$ even $\quad k \in \mathbb{N}$

n odd $\Longrightarrow n^k$ odd

n even $\Rightarrow n = 2m \wedge m \in \mathbb{Z} \Rightarrow n^k = (2m)^k = 2^k m^k \Longrightarrow$

$n^k = 2\left(2^{k-1} m^k\right) \wedge 2^{k-1} m^k \in \mathbb{Z}$ (since $k > 0$)

$\Longrightarrow n^k$ even.

n odd $\Rightarrow n = 2m + 1 \wedge m \in \mathbb{Z} \Rightarrow n^k = (2m+1)^k \Longrightarrow$

$$n^k = \binom{k}{0}(2m)^k + \binom{k}{1}(2m)^{k-1} + \cdots + \binom{k}{k-1}(2m)^1 + \underbrace{\binom{k}{k}(2m)^0}_{1}$$

$$= \underbrace{2\left[\binom{k}{0}(\;) + \binom{k}{1}(\;) + \cdots + \binom{k}{k-1}(\;)\right]}_{\in \mathbb{Z}} + 1$$

$\Longrightarrow n^k$ odd.

Every odd is the difference of two squares.

Prove:

$$n \text{ odd} \implies n = a^2 - b^2 \quad \text{where } a, b \in \mathbb{Z}$$

$$n \text{ odd} \implies n = 2k+1 = \underbrace{(k+1)^2}_{k^2+2k+1} - k^2.$$

Example: $11 = 2 \times 5 + 1 = 6^2 - 5^2 = 36 - 25.$

Direct Proofs on sets

To Prove that $S = T$ prove

- $S \subset T$ (S is a subset of T)
- $T \subset S$ (T is a subset of S)

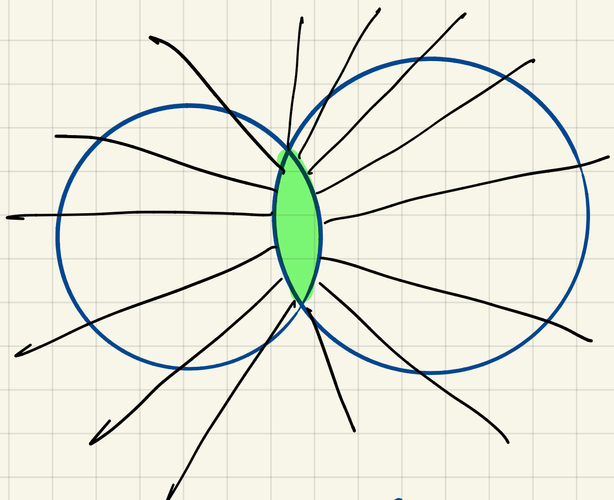
Proof looks likes

$$\forall x.\ x \in S \Rightarrow \ldots \Rightarrow x \in T$$

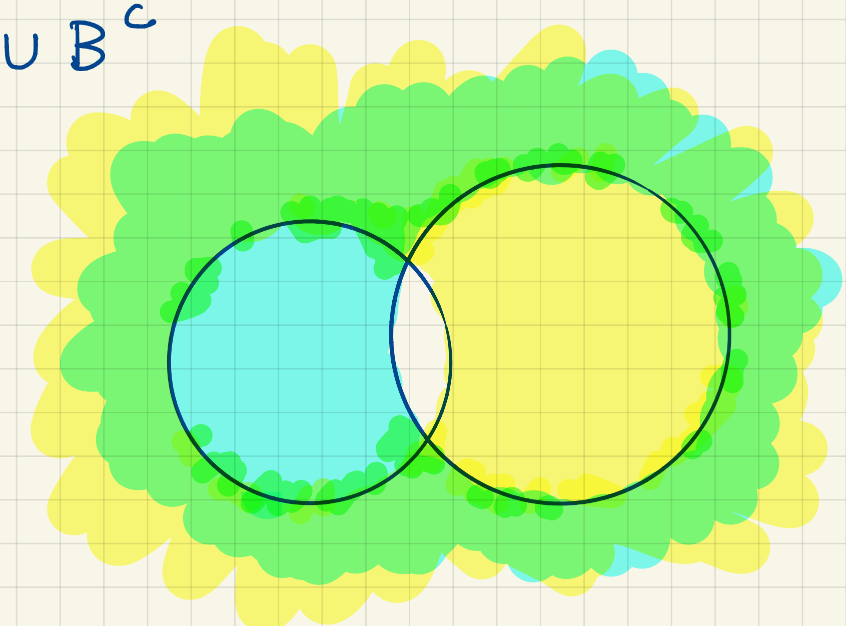
$$\forall x.\ x \in T \Rightarrow \ldots \Rightarrow x \in S$$

Example: Let $S^c = \{x : x \notin S\}$ (the complement)

Prove $(A \cap B)^c = A^c \cup B^c$



$(A \cap B)^c$



$A^c \cup B^c$

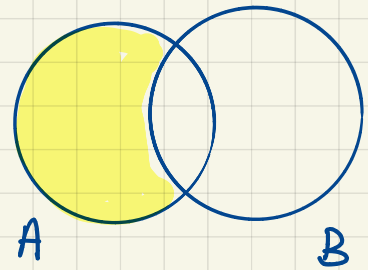
$$\underline{x \in (A \cap B)^c} \Rightarrow x \notin (A \cap B) \Rightarrow \neg(x \in (A \cap B)) \Rightarrow$$

$$\neg(x \in A \wedge x \in B) \underset{\text{D.M.}}{\Rightarrow} \neg(x \in A) \vee \neg(x \in B)$$

$$\Rightarrow x \notin A \vee x \notin B \Rightarrow x \in A^c \vee x \in B^c \Rightarrow \underline{x \in A^c \cup B^c}$$

(Direction of implications can be reversed).

Example: Let $A-B = \{x : x\in A \wedge x\notin B\}$



Prove: $(A-B)\cup(B-A) = (A\cup B)-(A\cap B)$

$$x\in (A-B)\cup(B-A) \Rightarrow x\in(A-B) \vee x\in(B-A)$$

$$\Rightarrow (x\in A \wedge x\notin B)\vee(x\in B\wedge x\notin A)$$

$$\Rightarrow (x\in A\vee x\in B)\wedge(x\in A\vee x\notin A)\wedge(x\notin B\vee x\in B)\wedge(x\notin B\vee x\notin A)$$

Distributivity $A\vee(B\wedge C) = (A\vee B)\wedge(A\vee C)$

$$\Rightarrow (x\in A\cup B)\wedge((\neg x\in B)\vee\neg(x\in A))$$

$$\xRightarrow[D.M.]{} (x\in A\cup B)\wedge\neg(x\in B\wedge x\in A)\Rightarrow$$

$$\Rightarrow (x\in A\cup B)\wedge\neg(x\in A\cap B)\Rightarrow(x\in A\cup B)\wedge x\notin(A\cap B)\Rightarrow$$

$$x\in(A\cup B)-(A\cap B)$$