A formula from the past ...

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

And systems of Linear Equations !

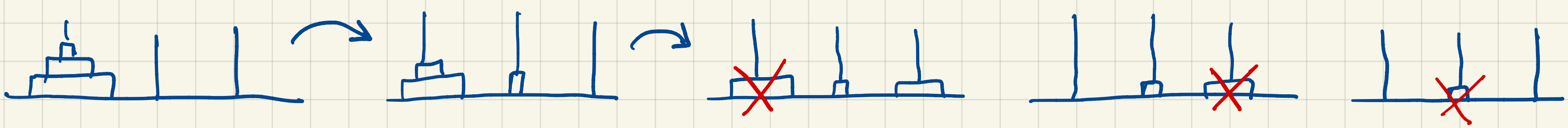
More on recurrences

Consider the following variant of Tower of Hanoi.

When the largest present disk is uncovered, it explodes!

The goal is to make all disks disappear.

Example:

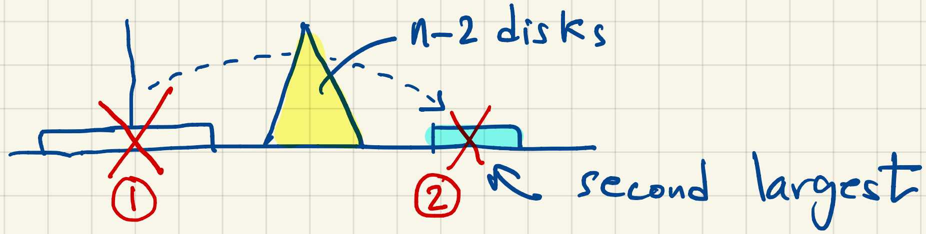


$n=3$: A total of 2 moves. $a_3 = 2$

$n=2$: $a_2 = 1$

$n=1$: $a_1 = 0$

Key idea: when largest disk uncovered, we have the following:



$$a_n = \underbrace{[2^{n-2}-1]}_{\text{move } n-2 \text{ disks}} + \underbrace{1}_{\text{uncover largest}} + \underbrace{a_{n-2}}_{\text{Solve a problem with } n-2 \text{ disks}} = a_{n-2} + 2^{n-2}$$

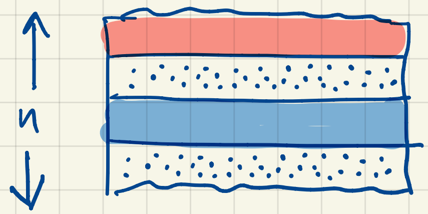
$$a_n = a_{n-2} + 2^{n-2} \qquad n \geqslant 3$$

$$a_1 = 0$$

$$a_2 = 1$$

what's a_0 ? $\quad (a_0 = 0)$

A flag.



Three colors: Blue, Red, White

n stripes

Adjacent stripes can't have same color

Question: How many possible flags?

1. Choose a color for 1^{st} stripe 3
2. Choose a color for 2^{nd} stripe 2
3. Choose a color for 3^{rd} stripe 2

⋮

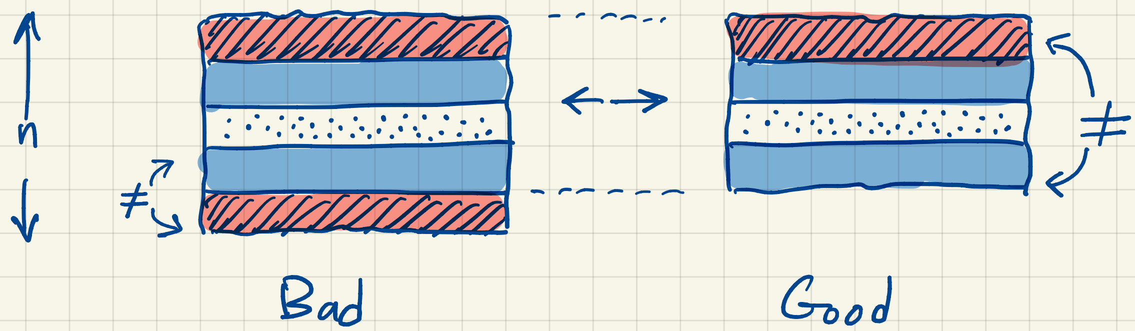
n. choose a color for n^{th} stripe 2

$$3 \times 2^{n-1}$$

Good flag: 1^{st} & n^{th} stripes have different colors (additional condition)

Let a_n, b_n be # good/bad flags: $a_n + b_n = 3 \times 2^{n-1}$

Key observation: $b_n = a_{n-1}$ $\quad (n \geqslant 3$ in pic below$)$



$(\text{also } b_2 = a_1 = 0)$

$$a_n + a_{n-1} = 3 \times 2^{n-1} \Rightarrow a_n = 3 \times 2^{n-1} - a_{n-1} \quad (n \geqslant 2)$$

Example: $a_1 = 0$

$a_2 = 3 \times 2^{2-1} - 0 = 6$

$a_3 = 3 \times 2^{3-1} - 6 = 6$

$a_4 = 3 \times 2^{4-1} - 6 = 18$

$a_5 = 3 \times 2^{5-1} - 18 = 30$

$\vdots$

Two recurrences :

$$a_n = a_{n-2} + 2^{n-2}$$

$$a_n = 3 \times 2^{n-1} - a_{n-1}$$

They are different than

$$a_n = \sum_{i<n} k_i a_i = k_{n-1} a_{n-1} + k_{n-2} a_{n-2} + \cdots \quad [\text{linear homogeneous}]$$

Why Linear homogeneous ?

Because we can solve them systematically !

Transform to homogeneous:

$$a_n = a_{n-2} + 2^{n-2}$$ ← eliminate this

$\ominus$

$$2a_{n-1} = 2a_{n-3} + 2 \cdot 2^{n-3}$$

$$a_n - 2a_{n-1} = a_{n-2} - 2a_{n-3} + \cancel{(2^{n-2} - 2^{n-2})}$$

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3} \qquad (n \geq 3)$$

since a_0 defined

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$$a_n = 3 \times 2^{n-1} - a_{n-1}$$

$\ominus$

$$2a_{n-1} = 2 \times 3 \times 2^{n-2} - 2 \cdot a_{n-2}$$

$$a_n - 2a_{n-1} = -a_{n-1} + 2a_{n-2}$$

$$a_n = a_{n-1} + 2a_{n-2} \qquad (n \geq 3)$$

Example:

Eliminate once

$$R_n = R_{n-1} + n$$

$$\circleddash \quad R_{n-1} = R_{n-2} + n-1$$

$$R_n - R_{n-1} = R_{n-1} - R_{n-2} + 1$$

Eliminate again.

$$R_n = 2R_{n-1} - R_{n-2} + 1$$

$$\circleddash \quad R_{n-1} = 2R_{n-2} - R_{n-3} + 1$$

$$R_n - R_{n-1} = 2R_{n-1} - R_{n-2} - 2R_{n-2} + R_{n-3}$$

$$\boxed{R_n = 3R_{n-1} - 3R_{n-2} + R_{n-3}} \quad (n \geqslant 3)$$

since R_0 defined

Solve for $a_n = A a_{n-1} + B a_{n-2}$

Wishful thinking: Say $a_n = p^n$ for some p.

$$p^n = A p^{n-1} + B p^{n-2}$$

$p^{n-2}(p^2 - Ap - B) = 0$ Ignore the trivial sol. $p = 0$

Characteristic equation

Find p by solving $x^2 - Ax - B = 0$ $\to p, q$

But $a_n = p^n$ is not necessarily satisfying base case(s).

We can show: ... (next page)

$\underline{p \neq q}$: $$a_n = c_1 p^n + c_2 q^n$$

$\underline{p = q}$: $$a_n = c_1 p^n + c_2 n p^n$$

Solve for c_1 and c_2 by satisfying base cases.

<u>Example 1</u>

$$a_n = \underbrace{a_{n-1}}_{A=1} + \underbrace{2a_{n-2}}_{B=2} \qquad a_1 = 0 \quad a_2 = 6$$

$$x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0 \Rightarrow x = \frac{1 \pm \sqrt{1+8}}{2} \begin{cases} \nearrow 2 \\ \searrow -1 \end{cases}$$

$$a_n = c_1 2^n + c_2 (-1)^n$$

$$\left.\begin{aligned} a_1 &= 2c_1 - c_2 = 0 \\ a_2 &= 4c_1 + c_2 = 6 \end{aligned}\right\} 6c_1 = 6 \Rightarrow c_1 = 1$$

$$c_2 = 2$$

$$a_n = 2^n + 2(-1)^n$$

Example 2:

$$a_n = 8a_{n-1} - 16a_{n-2} \qquad a_0 = 1 \qquad a_1 = 12$$

$$A = 8 \qquad B = -16$$

Characteristic Eq: $x^2 - Ax - B = 0$

$$x^2 - 8x + 16 = 0 \qquad x = \frac{8 \pm \sqrt{64 - 4 \times 1 \times 16}}{2} = \frac{8 \pm 0}{2} \begin{matrix} \nearrow 4 \\ \searrow 4 \end{matrix}$$

$p = q = 4.$

$$a_n = c_1 4^n + c_2 n 4^n$$

$$a_0 = c_1 + \underbrace{c_2 \times 0 \times 4^0}_{0} = 1 \Rightarrow c_1 = 1$$

$$a_1 = 4c_1 + c_2 \times 1 \times 4 = 12$$

$$4c_1 + 4c_2 = 12 \Rightarrow 4 + 4c_2 = 12 \Rightarrow c_2 = 2$$

$$a_n = 4^n + 2n4^n = 4^n(1 + 2n)$$

- What if $\underline{a_n = A a_{n-1}} + 0 \cdot a_{n-1}$

  Then $x^2 = Ax + 0 \Rightarrow x = A$ and $a_n = cA^n$

  (use one base case to find c)

- What if $a_n = B a_{n-2}$

  Then $x^2 = 0 \cdot x + B$ $\nearrow p = \sqrt{B}$, $\searrow q = -\sqrt{B}$

  $a_n = c_1 (\sqrt{B})^n + c_2 (-\sqrt{B})^n$

  (use two base cases to find c_1 and c_2)

Example 3. $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$

Characteristic Eq: $x^3 = 2x^2 + x - 2 \implies x^3 - 2x^2 - x + 2 = 0$

$p \quad q \quad r$

How to solve? Guess! $x=1$ works.

$a_n = c_1 p^n + c_2 q^n + c_3 r^n$

Rewrite:

$$(x-1)(x^2 + ax - 2) = 0$$

$+2$

$+x^3$

$a = -1$ since $ax^2 - x^2 = -2x^2$

Solve $x^2 - x - 2 = 0$ $\nearrow 2$, $\searrow -1$ (we just did)

$$a_n = c_1(1)^n + c_2\, 2^n + c_3\,(-1)^n$$

$$a_n = c_1 + c_2\, 2^n + c_3\,(-1)^n$$

$$\begin{aligned}
a_0 &= c_1 + c_2 + c_3 = 0 \\
a_1 &= c_1 + 2c_2 - c_3 = 0 \\
a_2 &= c_1 + 4c_2 + c_3 = 1
\end{aligned}$$

$$2c_1 + 3c_2 = 0$$

$$2c_1 + 6c_2 = 1$$

$$2c_1 = -1$$

$$c_1 = -\frac{1}{2}$$

$$c_1 = -\frac{1}{2} \qquad c_2 = \frac{1}{3} \qquad c_3 = \frac{1}{6}$$

$$a_n = -\frac{1}{2} + \frac{1}{3}\,2^n + \frac{1}{6}(-1)^n$$

$$a_n = \frac{2^{n+1} - 3 + (-1)^n}{6}$$

$$a_n = Aa_{n-1} + Ba_{n-2} + Ca_{n-3}$$

When solving for $x^3 - Ax^2 - Bx - C = 0 \to p, q, r$

<u>p, q, r all different:</u>

$$c_1 p^n + c_2 q^n + c_3 r^n$$

<u>$p = q \neq r$:</u>

$$\underbrace{c_1 p^n + c_2 n p^n} + c_3 r^n$$

<u>$p = q = r$:</u>

$$\underbrace{c_1 p^n + c_2 n p^n + c_3 n^2 p^n}$$